FLUID MECHANICS

12

? This shark must swim constantly to keep from sinking to the bottom of the ocean, yet the orange tropical fish can remain at the same level in the water with little effort. Why is there a difference?

LEARNING GOALS

By studying this chapter, you will learn:

- The meaning of the density of a material and the average density of a body.
- What is meant by the pressure in a fluid, and how it is measured.
- How to calculate the buoyant force that a fluid exerts on a body immersed in it.
- The significance of laminar versus turbulent fluid flow, and how the speed of flow in a tube depends on the tube size.
- How to use Bernoulli's equation to relate pressure and flow speed at different points in certain types of flow.

Fluids play a vital role in many aspects of everyday life. We drink them, breathe them, swim in them. They circulate through our bodies and control our weather. Airplanes fly through them; ships float in them. A fluid is any substance that can flow; we use the term for both liquids and gases. We usually think of a gas as easily compressed and a liquid as nearly incompressible, although there are exceptional cases.

We begin our study with **fluid statics,** the study of fluids at rest in equilibrium situations. Like other equilibrium situations, it is based on Newton's first and third laws. We will explore the key concepts of density, pressure, and buoyancy. **Fluid dynamics,** the study of fluids in motion, is much more complex; indeed, it is one of the most complex branches of mechanics. Fortunately, we can analyze many important situations using simple idealized models and familiar principles such as Newton's laws and conservation of energy. Even so, we will barely scratch the surface of this broad and interesting topic.

12.1 Density

An important property of any material is its **density,** defined as its mass per unit volume. A homogeneous material such as ice or iron has the same density throughout. We use ρ (the Greek letter rho) for density. If a mass m of homogeneous material has volume V, the density ρ is

$$\rho = \frac{m}{V} \quad \text{(definition of density)} \tag{12.1}$$

Two objects made of the same material have the same density even though they may have different masses and different volumes. That's because the *ratio* of mass to volume is the same for both objects (Fig. 12.1).

12.1 Two objects with different masses and different volumes but the same density.

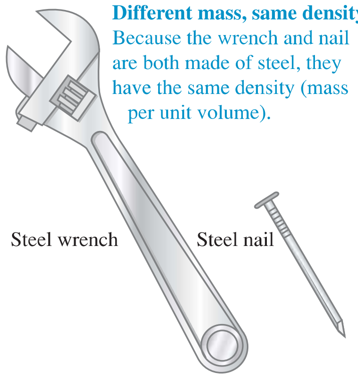


Table 12.1 Densities of Some Common Substances

Material	Density (kg/m^3)*	Material	Density (kg/m^3)*
Air (1 atm, 20°C)	1.20	Iron, steel	7.8×10^3
Ethanol	0.81×10^3	Brass	8.6×10^3
Benzene	0.90×10^3	Copper	8.9×10^3
Ice	0.92×10^3	Silver	10.5×10^3
Water	1.00×10^3	Lead	11.3×10^3
Seawater	1.03×10^3	Mercury	13.6×10^3
Blood	1.06×10^3	Gold	19.3×10^3
Glycerine	1.26×10^3	Platinum	21.4×10^3
Concrete	2×10^3	White dwarf star	10^{10}
Aluminum	2.7×10^3	Neutron star	10^{18}

*To obtain the densities in grams per cubic centimeter, simply divide by 10^3.

The SI unit of density is the kilogram per cubic meter ($1\ kg/m^3$). The cgs unit, the gram per cubic centimeter ($1\ g/cm^3$), is also widely used:

$$1\ g/cm^3 = 1000\ kg/m^3$$

The densities of some common substances at ordinary temperatures are given in Table 12.1. Note the wide range of magnitudes. The densest material found on earth is the metal osmium ($\rho = 22{,}500\ kg/m^3$), but its density pales by comparison to the densities of exotic astronomical objects such as white dwarf stars and neutron stars.

The **specific gravity** of a material is the ratio of its density to the density of water at 4.0°C, $1000\ kg/m^3$; it is a pure number without units. For example, the specific gravity of aluminum is 2.7. "Specific gravity" is a poor term, since it has nothing to do with gravity; "relative density" would have been better.

The density of some materials varies from point to point within the material. One example is the material of the human body, which includes low-density fat (about $940\ kg/m^3$) and high-density bone (from 1700 to $2500\ kg/m^3$). Two others are the earth's atmosphere (which is less dense at high altitudes) and oceans (which are denser at greater depths). For these materials, Eq. (12.1) describes the **average density.** In general, the density of a material depends on environmental factors such as temperature and pressure.

Measuring density is an important analytical technique. For example, we can determine the charge condition of a storage battery by measuring the density of its electrolyte, a sulfuric acid solution. As the battery discharges, the sulfuric acid (H_2SO_4) combines with lead in the battery plates to form insoluble lead sulfate ($PbSO_4$), decreasing the concentration of the solution. The density decreases from about $1.30 \times 10^3\ kg/m^3$ for a fully charged battery to $1.15 \times 10^3\ kg/m^3$ for a discharged battery.

Another automotive example is permanent-type antifreeze, which is usually a solution of ethylene glycol ($\rho = 1.12 \times 10^3\ kg/m^3$) and water. The freezing point of the solution depends on the glycol concentration, which can be determined by measuring the specific gravity. Such measurements can be performed by using a device called a hydrometer, which we'll discuss in Section 12.3.

Example 12.1 The weight of a roomful of air

Find the mass and weight of the air at 20°C in a living room with a 4.0 m × 5.0 m floor and a ceiling 3.0 m high, and the mass and weight of an equal volume of water.

SOLUTION

IDENTIFY and SET UP: We assume that the air density is the same throughout the room. (Air is less dense at high elevations than near

sea level, but the density varies negligibly over the room's 3.0-m height; see Section 12.2.) We use Eq. (12.1) to relate the mass m_{air} to the room's volume V (which we'll calculate) and the air density ρ_{air} (given in Table 12.1).

EXECUTE: We have $V = (4.0\text{ m})(5.0\text{ m})(3.0\text{ m}) = 60\text{ m}^3$, so from Eq. (12.1),

$$m_{\text{air}} = \rho_{\text{air}}V = (1.20\text{ kg/m}^3)(60\text{ m}^3) = 72\text{ kg}$$
$$w_{\text{air}} = m_{\text{air}}g = (72\text{ kg})(9.8\text{ m/s}^2) = 700\text{ N} = 160\text{ lb}$$

The mass and weight of an equal volume of water are

$$m_{\text{water}} = \rho_{\text{water}}V = (1000\text{ kg/m}^3)(60\text{ m}^3) = 6.0 \times 10^4\text{ kg}$$
$$w_{\text{water}} = m_{\text{water}}g = (6.0 \times 10^4\text{ kg})(9.8\text{ m/s}^2)$$
$$= 5.9 \times 10^5\text{ N} = 1.3 \times 10^5\text{ lb} = 66\text{ tons}$$

EVALUATE: A roomful of air weighs about the same as an average adult. Water is nearly a thousand times denser than air, so its mass and weight are larger by the same factor. The weight of a roomful of water would collapse the floor of an ordinary house.

Test Your Understanding of Section 12.1 Rank the following objects in order from highest to lowest average density: (i) mass 4.00 kg, volume $1.60 \times 10^{-3}\text{ m}^3$; (ii) mass 8.00 kg, volume $1.60 \times 10^{-3}\text{ m}^3$; (iii) mass 8.00 kg, volume $3.20 \times 10^{-3}\text{ m}^3$; (iv) mass 2560 kg, volume 0.640 m^3; (v) mass 2560 kg, volume 1.28 m^3.

12.2 Pressure in a Fluid

When a fluid (either liquid or gas) is at rest, it exerts a force perpendicular to any surface in contact with it, such as a container wall or a body immersed in the fluid. This is the force that you feel pressing on your legs when you dangle them in a swimming pool. While the fluid as a whole is at rest, the molecules that make up the fluid are in motion; the force exerted by the fluid is due to molecules colliding with their surroundings.

If we think of an imaginary surface *within* the fluid, the fluid on the two sides of the surface exerts equal and opposite forces on the surface. (Otherwise, the surface would accelerate and the fluid would not remain at rest.) Consider a small surface of area dA centered on a point in the fluid; the normal force exerted by the fluid on each side is $dF_{\perp}$ (Fig. 12.2). We define the **pressure** p at that point as the normal force per unit area—that is, the ratio of $dF_{\perp}$ to dA (Fig. 12.3):

$$p = \frac{dF_{\perp}}{dA} \quad \text{(definition of pressure)} \tag{12.2}$$

If the pressure is the same at all points of a finite plane surface with area A, then

$$p = \frac{F_{\perp}}{A} \tag{12.3}$$

where $F_{\perp}$ is the net normal force on one side of the surface. The SI unit of pressure is the **pascal,** where

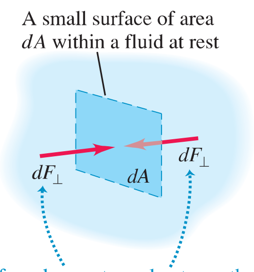
$$1\text{ pascal} = 1\text{ Pa} = 1\text{ N/m}^2$$

We introduced the pascal in Chapter 11. Two related units, used principally in meteorology, are the *bar,* equal to 10^5 Pa, and the *millibar,* equal to 100 Pa.

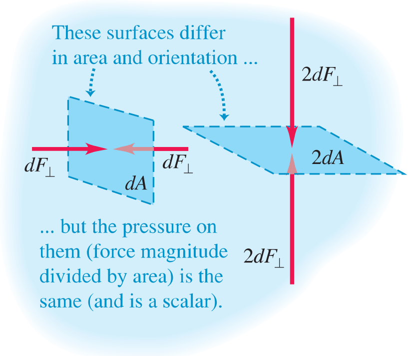
Atmospheric pressure p_a is the pressure of the earth's atmosphere, the pressure at the bottom of this sea of air in which we live. This pressure varies with weather changes and with elevation. Normal atmospheric pressure at sea level (an average value) is 1 *atmosphere* (atm), defined to be exactly 101,325 Pa. To four significant figures,

$$(p_a)_{\text{av}} = 1\text{ atm} = 1.013 \times 10^5\text{ Pa}$$
$$= 1.013\text{ bar} = 1013\text{ millibar} = 14.70\text{ lb/in.}^2$$

12.2 Forces acting on a small surface within a fluid at rest.



12.3 The pressure on either side of a surface is force divided by area. Pressure is a scalar with units of newtons per square meter. By contrast, force is a vector with units of newtons.



CAUTION **Don't confuse pressure and force** In everyday language the words "pressure" and "force" mean pretty much the same thing. In fluid mechanics, however, these words describe distinct quantities with different characteristics. Fluid pressure acts perpendicular to any surface in the fluid, no matter how that surface is oriented (Fig. 12.3). Hence pressure has no intrinsic direction of its own; it's a scalar. By contrast, force is a vector with a definite direction. Remember, too, that pressure is force per unit area. As Fig. 12.3 shows, a surface with twice the area has twice as much force exerted on it by the fluid, so the pressure is the same.

Example 12.2 The force of air

In the room described in Example 12.1, what is the total downward force on the floor due to an air pressure of 1.00 atm?

SOLUTION

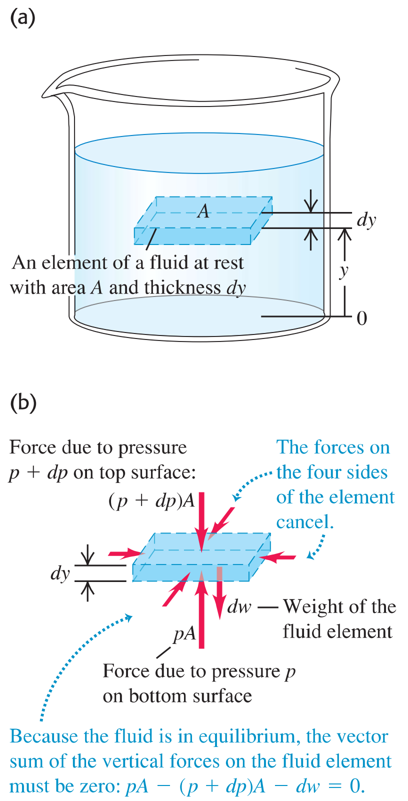
IDENTIFY and SET UP: This example uses the relationship among the pressure p of a fluid (air), the area A subjected to that pressure, and the resulting normal force $F_\perp$ the fluid exerts. The pressure is uniform, so we use Eq. (12.3), $F_\perp = pA$, to determine $F_\perp$. The floor is horizontal, so $F_\perp$ is vertical (downward).

EXECUTE: We have $A = (4.0\text{ m})(5.0\text{ m}) = 20\text{ m}^2$, so from Eq. (12.3),

$$\begin{aligned} F_\perp = pA &= (1.013 \times 10^5\text{ N/m}^2)(20\text{ m}^2) \\ &= 2.0 \times 10^6\text{ N} = 4.6 \times 10^5\text{ lb} = 230\text{ tons} \end{aligned}$$

EVALUATE: Unlike the water in Example 12.1, $F_\perp$ will not collapse the floor here, because there is an *upward* force of equal magnitude on the floor's underside. If the house has a basement, this upward force is exerted by the air underneath the floor. In this case, if we neglect the thickness of the floor, the *net* force due to air pressure is zero.

12.4 The forces on an element of fluid in equilibrium.



Pressure, Depth, and Pascal's Law

If the weight of the fluid can be neglected, the pressure in a fluid is the same throughout its volume. We used that approximation in our discussion of bulk stress and strain in Section 11.4. But often the fluid's weight is *not* negligible. Atmospheric pressure is less at high altitude than at sea level, which is why an airplane cabin has to be pressurized when flying at 35,000 feet. When you dive into deep water, your ears tell you that the pressure increases rapidly with increasing depth below the surface.

We can derive a general relationship between the pressure p at any point in a fluid at rest and the elevation y of the point. We'll assume that the density ρ has the same value throughout the fluid (that is, the density is *uniform*), as does the acceleration due to gravity g. If the fluid is in equilibrium, every volume element is in equilibrium. Consider a thin element of fluid with thickness dy (Fig. 12.4a). The bottom and top surfaces each have area A, and they are at elevations y and $y + dy$ above some reference level where $y = 0$. The volume of the fluid element is $dV = A\,dy$, its mass is $dm = \rho\,dV = \rho A\,dy$, and its weight is $dw = dm\,g = \rho g A\,dy$.

What are the other forces on this fluid element (Fig 12.4b)? Let's call the pressure at the bottom surface p; then the total y-component of upward force on this surface is pA. The pressure at the top surface is $p + dp$, and the total y-component of (downward) force on the top surface is $-(p + dp)A$. The fluid element is in equilibrium, so the total y-component of force, including the weight and the forces at the bottom and top surfaces, must be zero:

$$\sum F_y = 0 \qquad \text{so} \qquad pA - (p + dp)A - \rho g A\,dy = 0$$

When we divide out the area A and rearrange, we get

$$\frac{dp}{dy} = -\rho g \tag{12.4}$$

This equation shows that when y increases, p decreases; that is, as we move upward in the fluid, pressure decreases, as we expect. If p_1 and p_2 are the pressures at elevations y_1 and y_2, respectively, and if ρ and g are constant, then

$$p_2 - p_1 = -\rho g(y_2 - y_1) \qquad \text{(pressure in a fluid of uniform density)} \qquad (12.5)$$

It's often convenient to express Eq. (12.5) in terms of the *depth* below the surface of a fluid (Fig. 12.5). Take point 1 at any level in the fluid and let p represent the pressure at this point. Take point 2 at the *surface* of the fluid, where the pressure is p_0 (subscript zero for zero depth). The depth of point 1 below the surface is $h = y_2 - y_1$, and Eq. (12.5) becomes

$$p_0 - p = -\rho g(y_2 - y_1) = -\rho g h \qquad \text{or}$$

$$p = p_0 + \rho g h \qquad \text{(pressure in a fluid of uniform density)} \qquad (12.6)$$

The pressure p at a depth h is greater than the pressure p_0 at the surface by an amount $\rho g h$. Note that the pressure is the same at any two points at the same level in the fluid. The *shape* of the container does not matter (Fig. 12.6).

Equation (12.6) shows that if we increase the pressure p_0 at the top surface, possibly by using a piston that fits tightly inside the container to push down on the fluid surface, the pressure p at any depth increases by exactly the same amount. This fact was recognized in 1653 by the French scientist Blaise Pascal (1623–1662) and is called *Pascal's law.*

Pascal's law: Pressure applied to an enclosed fluid is transmitted undiminished to every portion of the fluid and the walls of the containing vessel.

The hydraulic lift shown schematically in Fig. 12.7 illustrates Pascal's law. A piston with small cross-sectional area A_1 exerts a force F_1 on the surface of a liquid such as oil. The applied pressure $p = F_1/A_1$ is transmitted through the connecting pipe to a larger piston of area A_2. The applied pressure is the same in both cylinders, so

$$p = \frac{F_1}{A_1} = \frac{F_2}{A_2} \qquad \text{and} \qquad F_2 = \frac{A_2}{A_1}F_1 \qquad (12.7)$$

The hydraulic lift is a force-multiplying device with a multiplication factor equal to the ratio of the areas of the two pistons. Dentist's chairs, car lifts and jacks, many elevators, and hydraulic brakes all use this principle.

For gases the assumption that the density ρ is uniform is realistic only over short vertical distances. In a room with a ceiling height of 3.0 m filled with air of uniform density 1.2 kg/m^3, the difference in pressure between floor and ceiling, given by Eq. (12.6), is

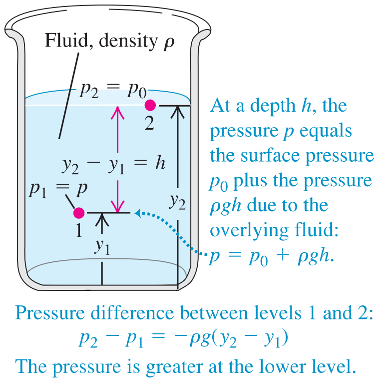
$$\rho g h = (1.2\ \text{kg/m}^3)(9.8\ \text{m/s}^2)(3.0\ \text{m}) = 35\ \text{Pa}$$

or about 0.00035 atm, a very small difference. But between sea level and the summit of Mount Everest (8882 m) the density of air changes by nearly a factor of 3, and in this case we cannot use Eq. (12.6). Liquids, by contrast, are nearly incompressible, and it is usually a very good approximation to regard their density as independent of pressure. A pressure of several hundred atmospheres will cause only a few percent increase in the density of most liquids.

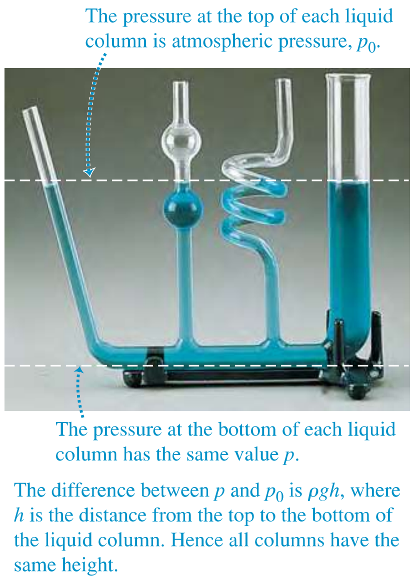
Absolute Pressure and Gauge Pressure

If the pressure inside a car tire is equal to atmospheric pressure, the tire is flat. The pressure has to be *greater* than atmospheric to support the car, so the significant quantity is the *difference* between the inside and outside pressures. When we say that the pressure in a car tire is "32 pounds" (actually 32 lb/in.^{2}, equal to 220 kPa or 2.2×10^5 Pa), we mean that it is *greater* than atmospheric pressure

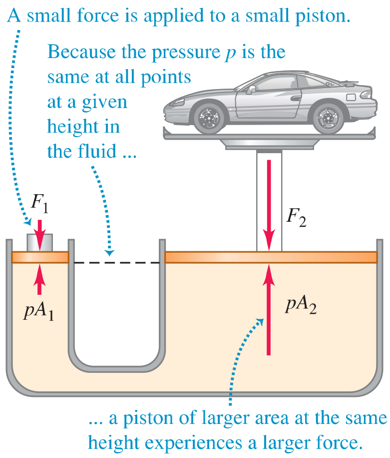
12.5 How pressure varies with depth in a fluid with uniform density.



12.6 Each fluid column has the same height, no matter what its shape.



12.7 The hydraulic lift is an application of Pascal's law. The size of the fluid-filled container is exaggerated for clarity.



(14.7 lb/in.^{2} or 1.01×10^5 Pa) by this amount. The *total* pressure in the tire is then 47 lb/in.^{2} or 320 kPa. The excess pressure above atmospheric pressure is usually called **gauge pressure,** and the total pressure is called **absolute pressure.** Engineers use the abbreviations psig and psia for "pounds per square inch gauge" and "pounds per square inch absolute," respectively. If the pressure is *less* than atmospheric, as in a partial vacuum, the gauge pressure is negative.

Example 12.3 Finding absolute and gauge pressures

Water stands 12.0 m deep in a storage tank whose top is open to the atmosphere. What are the absolute and gauge pressures at the bottom of the tank?

SOLUTION

IDENTIFY and SET UP: Table 11.2 indicates that water is nearly incompressible, so we can treat it as having uniform density. The level of the top of the tank corresponds to point 2 in Fig. 12.5, and the level of the bottom of the tank corresponds to point 1. Our target variable is p in Eq. (12.6). We have $h = 12.0$ m and $p_0 = 1$ atm $= 1.01 \times 10^5$ Pa.

EXECUTE: From Eq. (12.6), the pressures are

absolute:

$$\begin{aligned} p &= p_0 + \rho g h \\ &= (1.01 \times 10^5 \text{ Pa}) + (1000 \text{ kg/m}^3)(9.80 \text{ m/s}^2)(12.0 \text{ m}) \\ &= 2.19 \times 10^5 \text{ Pa} = 2.16 \text{ atm} = 31.8 \text{ lb/in.}^2 \end{aligned}$$

gauge:

$$\begin{aligned} p - p_0 &= (2.19 - 1.01) \times 10^5 \text{ Pa} \\ &= 1.18 \times 10^5 \text{ Pa} = 1.16 \text{ atm} = 17.1 \text{ lb/in.}^2 \end{aligned}$$

EVALUATE: A pressure gauge at the bottom of such a tank would probably be calibrated to read gauge pressure rather than absolute pressure.

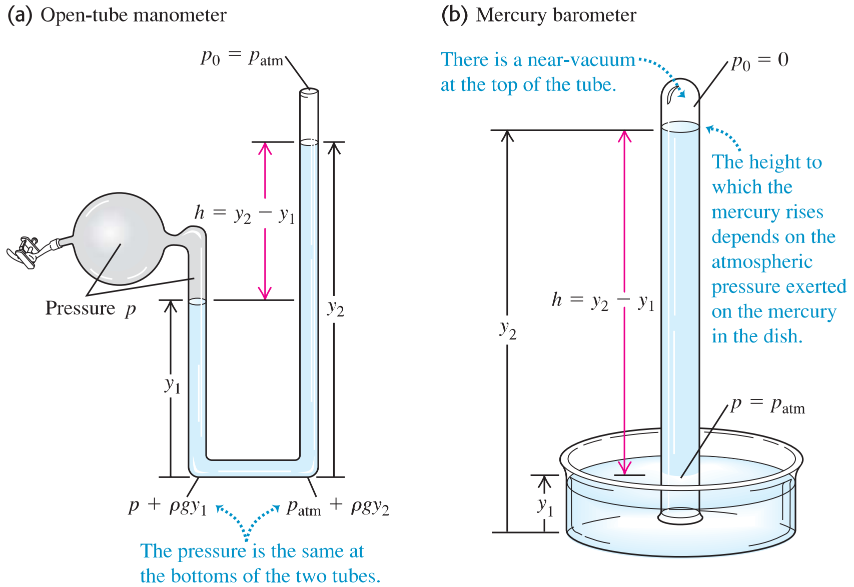
Pressure Gauges

The simplest pressure gauge is the open-tube *manometer* (Fig. 12.8a). The U-shaped tube contains a liquid of density ρ, often mercury or water. The left end of the tube is connected to the container where the pressure p is to be measured, and the right end is open to the atmosphere at pressure $p_0 = p_{\text{atm}}$. The pressure at the bottom of the tube due to the fluid in the left column is $p + \rho g y_1$, and the pressure at the bottom due to the fluid in the right column is $p_{\text{atm}} + \rho g y_2$. These pressures are measured at the same level, so they must be equal:

$$\begin{aligned} p + \rho g y_1 &= p_{\text{atm}} + \rho g y_2 \\ p - p_{\text{atm}} &= \rho g (y_2 - y_1) = \rho g h \end{aligned} \tag{12.8}$$

In Eq. (12.8), p is the *absolute pressure,* and the difference $p - p_{\text{atm}}$ between absolute and atmospheric pressure is the gauge pressure. Thus the gauge pressure is proportional to the difference in height $h = y_2 - y_1$ of the liquid columns.

12.8 Two types of pressure gauge.



Another common pressure gauge is the **mercury barometer.** It consists of a long glass tube, closed at one end, that has been filled with mercury and then inverted in a dish of mercury (Fig. 12.8b). The space above the mercury column contains only mercury vapor; its pressure is negligibly small, so the pressure p_0 at the top of the mercury column is practically zero. From Eq. (12.6),

$$p_{atm} = p = 0 + \rho g(y_2 - y_1) = \rho g h \tag{12.9}$$

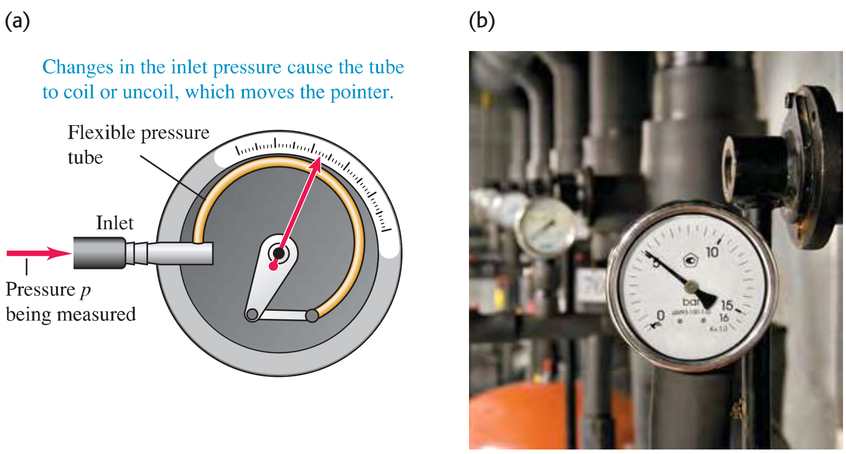
Thus the mercury barometer reads the atmospheric pressure p_{atm} directly from the height of the mercury column.

Pressures are often described in terms of the height of the corresponding mercury column, as so many "inches of mercury" or "millimeters of mercury" (abbreviated mm Hg). A pressure of 1 mm Hg is called *1 torr,* after Evangelista Torricelli, inventor of the mercury barometer. But these units depend on the density of mercury, which varies with temperature, and on the value of g, which varies with location, so the pascal is the preferred unit of pressure.

Many types of pressure gauges use a flexible sealed tube (Fig. 12.9). A change in the pressure either inside or outside the tube causes a change in its dimensions. This change is detected optically, electrically, or mechanically.

Application Gauge Pressure of Blood

Blood-pressure readings, such as 130/80, give the maximum and minimum gauge pressures in the arteries, measured in mm Hg or torr. Blood pressure varies with vertical position within the body; the standard reference point is the upper arm, level with the heart.



12.9 (a) A Bourdon pressure gauge. When the pressure inside the flexible tube increases, the tube straightens out a little, deflecting the attached pointer. (b) This Bourdon-type pressure gauge is connected to a high-pressure gas line. The gauge pressure shown is just over 5 bars (1 bar $= 10^5$ Pa).

Example 12.4 A tale of two fluids

A manometer tube is partially filled with water. Oil (which does not mix with water) is poured into the left arm of the tube until the oil–water interface is at the midpoint of the tube as shown. Both arms of the tube are open to the air. Find a relationship between the heights h_{oil} and h_{water}.

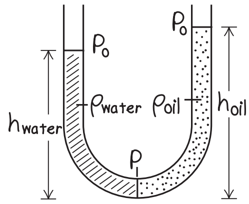
SOLUTION

IDENTIFY and SET UP: Figure 12.10 shows our sketch. The relationship between pressure and depth given by Eq. (12.6) applies only to fluids of uniform density; we have two fluids of different densities, so we must write a separate pressure–depth relationship for each. Both fluid columns have pressure p at the bottom (where they are in contact and in equilibrium) and are both at atmospheric pressure p_0 at the top (where both are in contact with and in equilibrium with the air).

EXECUTE: Writing Eq. (12.6) for each fluid gives

$$p = p_0 + \rho_{water} g h_{water}$$
$$p = p_0 + \rho_{oil} g h_{oil}$$

12.10 Our sketch for this problem.



Since the pressure p at the bottom of the tube is the same for both fluids, we set these two expressions equal to each other and solve for h_{oil} in terms of h_{water}. You can show that the result is

$$h_{oil} = \frac{\rho_{water}}{\rho_{oil}} h_{water}$$

EVALUATE: Water $(\rho_{water} = 1000\ \text{kg/m}^3)$ is denser than oil $(\rho_{oil} \approx 850\ \text{kg/m}^3)$, so h_{oil} is greater than h_{water} as Fig. 12.10 shows. It takes a greater height of low-density oil to produce the same pressure p at the bottom of the tube.

Test Your Understanding of Section 12.2 Mercury is less dense at high temperatures than at low temperatures. Suppose you move a mercury barometer from the cold interior of a tightly sealed refrigerator to outdoors on a hot summer day. You find that the column of mercury remains at the same height in the tube. Compared to the air pressure inside the refrigerator, is the air pressure outdoors (i) higher, (ii) lower, or (iii) the same? (Ignore the very small change in the dimensions of the glass tube due to the temperature change.)

12.3 Buoyancy

MasteringPHYSICS
PhET: Balloons & Buoyancy

Buoyancy is a familiar phenomenon: A body immersed in water seems to weigh less than when it is in air. When the body is less dense than the fluid, it floats. The human body usually floats in water, and a helium-filled balloon floats in air.

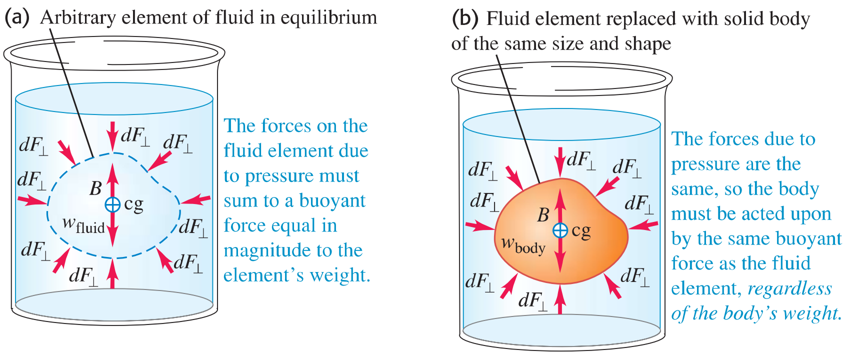
Archimedes's principle: When a body is completely or partially immersed in a fluid, the fluid exerts an upward force on the body equal to the weight of the fluid displaced by the body.

To prove this principle, we consider an arbitrary element of fluid at rest. In Fig. 12.11a the irregular outline is the surface boundary of this element of fluid. The arrows represent the forces exerted on the boundary surface by the surrounding fluid.

The entire fluid is in equilibrium, so the sum of all the y-components of force on this element of fluid is zero. Hence the sum of the y-components of the *surface* forces must be an upward force equal in magnitude to the weight mg of the fluid inside the surface. Also, the sum of the torques on the element of fluid must be zero, so the line of action of the resultant y-component of surface force must pass through the center of gravity of this element of fluid.

Now we remove the fluid inside the surface and replace it with a solid body having exactly the same shape (Fig. 12.11b). The pressure at every point is exactly the same as before. So the total upward force exerted on the body by the fluid is also the same, again equal in magnitude to the weight mg of the fluid displaced to make way for the body. We call this upward force the **buoyant force** on the solid body. The line of action of the buoyant force again passes through the center of gravity of the displaced fluid (which doesn't necessarily coincide with the center of gravity of the body).

When a balloon floats in equilibrium in air, its weight (including the gas inside it) must be the same as the weight of the air displaced by the balloon. A fish's flesh is denser than water, yet a fish can float while

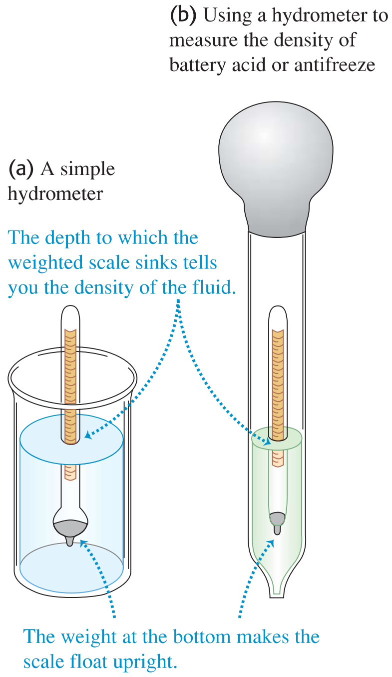


12.11 Archimedes's principle.

submerged because it has a gas-filled cavity within its body. This makes the fish's *average* density the same as water's, so its net weight is the same as the weight of the water it displaces. A body whose average density is *less* than that of a liquid can float partially submerged at the free upper surface of the liquid. The greater the density of the liquid, the less of the body is submerged. When you swim in seawater (density $1030\ \text{kg/m}^3$), your body floats higher than in fresh water ($1000\ \text{kg/m}^3$).

A practical example of buoyancy is the hydrometer, used to measure the density of liquids (Fig. 12.12a). The calibrated float sinks into the fluid until the weight of the fluid it displaces is exactly equal to its own weight. The hydrometer floats *higher* in denser liquids than in less dense liquids, and a scale in the top stem permits direct density readings. Figure 12.12b shows a type of hydrometer that is commonly used to measure the density of battery acid or antifreeze. The bottom of the large tube is immersed in the liquid; the bulb is squeezed to expel air and is then released, like a giant medicine dropper. The liquid rises into the outer tube, and the hydrometer floats in this sample of the liquid.

12.12 Measuring the density of a fluid.



Example 12.5 Buoyancy

A 15.0-kg solid gold statue is raised from the sea bottom (Fig. 12.13a). What is the tension in the hoisting cable (assumed massless) when the statue is (a) at rest and completely underwater and (b) at rest and completely out of the water?

SOLUTION

IDENTIFY and SET UP: In both cases the statue is in equilibrium and experiences three forces: its weight, the cable tension, and a buoyant force equal in magnitude to the weight of the fluid displaced by the statue (seawater in part (a), air in part (b)). Figure 12.13b shows the free-body diagram for the statue. Our target variables are the values of the tension in seawater (T_{sw}) and in air (T_{air}). We are given the mass m_{statue}, and we can calculate the buoyant force in seawater (B_{sw}) and in air (B_{air}) using Archimedes's principle.

EXECUTE: (a) To find B_{sw}, we first find the statue's volume V using the density of gold from Table 12.1:

$$V = \frac{m_{statue}}{\rho_{gold}} = \frac{15.0\ \text{kg}}{19.3 \times 10^3\ \text{kg/m}^3} = 7.77 \times 10^{-4}\ \text{m}^3$$

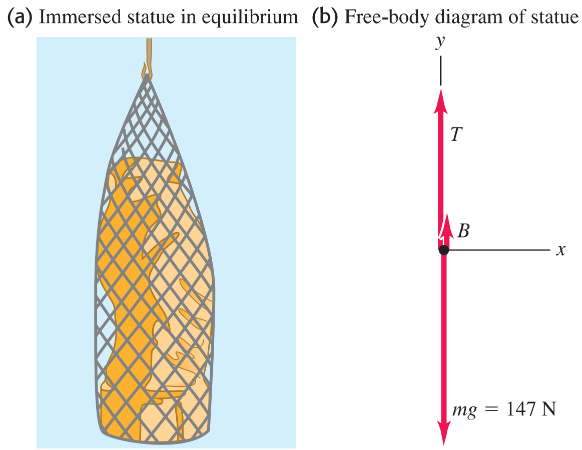
The buoyant force B_{sw} equals the weight of this same volume of seawater. Using Table 12.1 again:

$$\begin{aligned} B_{sw} &= w_{sw} = m_{sw}g = \rho_{sw}Vg \\ &= (1.03 \times 10^3\ \text{kg/m}^3)(7.77 \times 10^{-4}\ \text{m}^3)(9.80\ \text{m/s}^2) \\ &= 7.84\ \text{N} \end{aligned}$$

The statue is at rest, so the net external force acting on it is zero. From Fig. 12.13b,

$$\begin{aligned} \sum F_y &= B_{sw} + T_{sw} + (-m_{statue}g) = 0 \\ T_{sw} &= m_{statue}g - B_{sw} = (15.0\ \text{kg})(9.80\ \text{m/s}^2) - 7.84\ \text{N} \\ &= 147\ \text{N} - 7.84\ \text{N} = 139\ \text{N} \end{aligned}$$

12.13 What is the tension in the cable hoisting the statue?



A spring scale attached to the upper end of the cable will indicate a tension 7.84 N less than the statue's actual weight $m_{statue}g = 147\ \text{N}$.

(b) The density of air is about $1.2\ \text{kg/m}^3$, so the buoyant force of air on the statue is

$$\begin{aligned} B_{air} &= \rho_{air}Vg = (1.2\ \text{kg/m}^3)(7.77 \times 10^{-4}\ \text{m}^3)(9.80\ \text{m/s}^2) \\ &= 9.1 \times 10^{-3}\ \text{N} \end{aligned}$$

This is negligible compared to the statue's actual weight $m_{statue}g = 147\ \text{N}$. So within the precision of our data, the tension in the cable with the statue in air is $T_{air} = m_{statue}g = 147\ \text{N}$.

EVALUATE: Note that the buoyant force is proportional to the density of the *fluid* in which the statue is immersed, *not* the density of

Continued

the statue. The denser the fluid, the greater the buoyant force and the smaller the cable tension. If the fluid had the same density as the statue, the buoyant force would be equal to the statue's weight and the tension would be zero (the cable would go slack). If the fluid were denser than the statue, the tension would be *negative:* The buoyant force would be greater than the statue's weight, and a downward force would be required to keep the statue from rising upward.

12.14 The surface of the water acts like a membrane under tension, allowing this water strider to literally "walk on water."

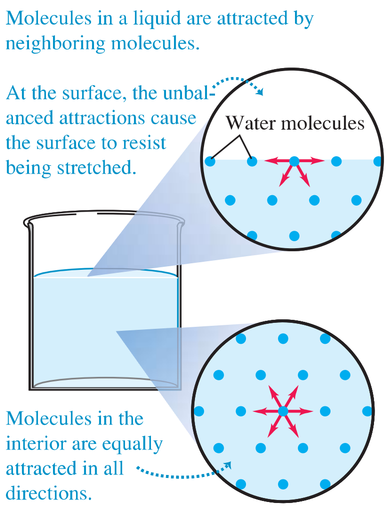
Surface Tension

An object less dense than water, such as an air-filled beach ball, floats with part of its volume below the surface. Conversely, a paper clip can rest *atop* a water surface even though its density is several times that of water. This is an example of **surface tension:** The surface of the liquid behaves like a membrane under tension (Fig. 12.14). Surface tension arises because the molecules of the liquid exert attractive forces on each other. There is zero net force on a molecule inside the volume of the liquid, but a surface molecule is drawn into the volume (Fig. 12.15). Thus the liquid tends to minimize its surface area, just as a stretched membrane does.

Surface tension explains why freely falling raindrops are spherical (*not* teardrop-shaped): A sphere has a smaller surface area for its volume than any other shape. It also explains why hot, soapy water is used for washing. To wash clothing thoroughly, water must be forced through the tiny spaces between the fibers (Fig. 12.16). To do so requires increasing the surface area of the water, which is difficult to achieve because of surface tension. The job is made easier by increasing the temperature of the water and adding soap, both of which decrease the surface tension.

Surface tension is important for a millimeter-sized water drop, which has a relatively large surface area for its volume. (A sphere of radius r has surface area $4\pi r^2$ and volume $(4\pi/3)r^3$. The ratio of surface area to volume is $3/r$, which increases with decreasing radius.) For large quantities of liquid, however, the ratio of surface area to volume is relatively small, and surface tension is negligible compared to pressure forces. For the remainder of this chapter, we will consider only fluids in bulk and hence will ignore the effects of surface tension.

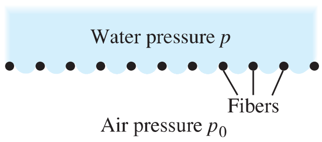
12.15 A molecule at the surface of a liquid is attracted into the bulk liquid, which tends to reduce the liquid's surface area.



Test Your Understanding of Section 12.3 You place a container of seawater on a scale and note the reading on the scale. You now suspend the statue of Example 12.5 in the water (Fig. 12.17). How does the scale reading change? (i) It increases by 7.84 N; (ii) it decreases by 7.84 N; (iii) it remains the same; (iv) none of these.

12.4 Fluid Flow

12.16 Surface tension makes it difficult to force water through small crevices. The required water pressure p can be reduced by using hot, soapy water, which has less surface tension.



We are now ready to consider *motion* of a fluid. Fluid flow can be extremely complex, as shown by the currents in river rapids or the swirling flames of a campfire. But some situations can be represented by relatively simple idealized models. An **ideal fluid** is a fluid that is *incompressible* (that is, its density cannot change) and has no internal friction (called **viscosity**). Liquids are approximately incompressible in most situations, and we may also treat a gas as incompressible if the pressure differences from one region to another are not too great. Internal friction in a fluid causes shear stresses when two adjacent layers of fluid move relative to each other, as when fluid flows inside a tube or around an obstacle. In some cases we can neglect these shear forces in comparison with forces arising from gravitation and pressure differences.

The path of an individual particle in a moving fluid is called a **flow line.** If the overall flow pattern does not change with time, the flow is called **steady flow.** In

steady flow, every element passing through a given point follows the same flow line. In this case the "map" of the fluid velocities at various points in space remains constant, although the velocity of a particular particle may change in both magnitude and direction during its motion. A **streamline** is a curve whose tangent at any point is in the direction of the fluid velocity at that point. When the flow pattern changes with time, the streamlines do not coincide with the flow lines. We will consider only steady-flow situations, for which flow lines and streamlines are identical.

The flow lines passing through the edge of an imaginary element of area, such as the area A in Fig. 12.18, form a tube called a **flow tube.** From the definition of a flow line, in steady flow no fluid can cross the side walls of a flow tube; the fluids in different flow tubes cannot mix.

Figure 12.19 shows patterns of fluid flow from left to right around three different obstacles. The photographs were made by injecting dye into water flowing between two closely spaced glass plates. These patterns are typical of **laminar flow,** in which adjacent layers of fluid slide smoothly past each other and the flow is steady. (A *lamina* is a thin sheet.) At sufficiently high flow rates, or when boundary surfaces cause abrupt changes in velocity, the flow can become irregular and chaotic. This is called **turbulent flow** (Fig. 12.20). In turbulent flow there is no steady-state pattern; the flow pattern changes continuously.

The Continuity Equation

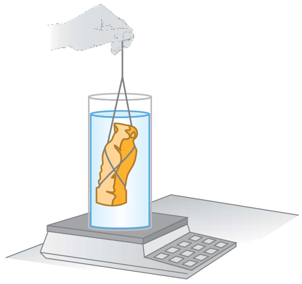
The mass of a moving fluid doesn't change as it flows. This leads to an important quantitative relationship called the **continuity equation.** Consider a portion of a flow tube between two stationary cross sections with areas A_1 and A_2 (Fig. 12.21). The fluid speeds at these sections are v_1 and v_2, respectively. No fluid flows in or out across the sides of the tube because the fluid velocity is tangent to the wall at every point on the wall. During a small time interval dt, the fluid at A_1 moves a distance $v_1\,dt$, so a cylinder of fluid with height $v_1\,dt$ and volume $dV_1 = A_1 v_1\,dt$ flows into the tube across A_1. During this same interval, a cylinder of volume $dV_2 = A_2 v_2\,dt$ flows out of the tube across A_2.

Let's first consider the case of an incompressible fluid so that the density ρ has the same value at all points. The mass dm_1 flowing into the tube across A_1 in time dt is $dm_1 = \rho A_1 v_1\,dt$. Similarly, the mass dm_2 that flows out across A_2 in the same time is $dm_2 = \rho A_2 v_2\,dt$. In steady flow the total mass in the tube is constant, so $dm_1 = dm_2$ and

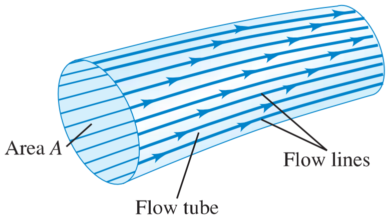
$$\rho A_1 v_1\,dt = \rho A_2 v_2\,dt \qquad \text{or}$$

$$A_1 v_1 = A_2 v_2 \qquad \text{(continuity equation, incompressible fluid)} \qquad (12.10)$$

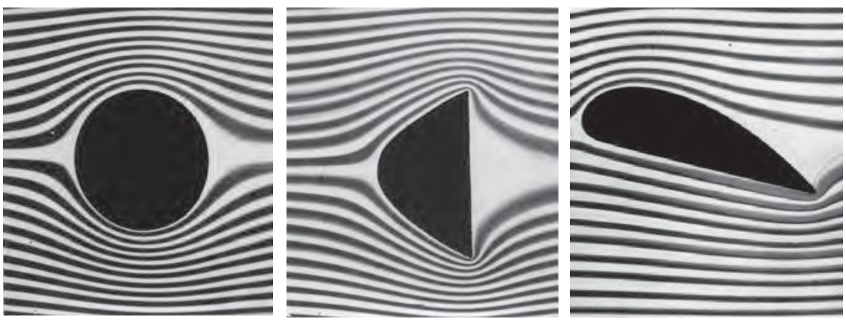
12.17 How does the scale reading change when the statue is immersed in water?



12.18 A flow tube bounded by flow lines. In steady flow, fluid cannot cross the walls of a flow tube.

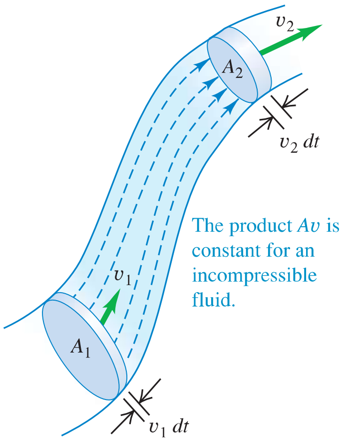


12.19 Laminar flow around obstacles of different shapes.



12.20 The flow of smoke rising from these incense sticks is laminar up to a certain point, and then becomes turbulent.

12.21 A flow tube with changing cross-sectional area. If the fluid is incompressible, the product Av has the same value at all points along the tube.



The product Av is the *volume flow rate* dV/dt, the rate at which volume crosses a section of the tube:

$$\frac{dV}{dt} = Av \qquad \text{(volume flow rate)} \tag{12.11}$$

The *mass* flow rate is the mass flow per unit time through a cross section. This is equal to the density ρ times the volume flow rate dV/dt.

Equation (12.10) shows that the volume flow rate has the same value at all points along any flow tube. When the cross section of a flow tube decreases, the speed increases, and vice versa. A broad, deep part of a river has larger cross section and slower current than a narrow, shallow part, but the volume flow rates are the same in both. This is the essence of the familiar maxim, "Still waters run deep." The stream of water from a faucet narrows as it gains speed during its fall, but dV/dt is the same everywhere along the stream. If a water pipe with 2-cm diameter is connected to a pipe with 1-cm diameter, the flow speed is four times as great in the 1-cm part as in the 2-cm part.

We can generalize Eq. (12.10) for the case in which the fluid is *not* incompressible. If ρ_1 and ρ_2 are the densities at sections 1 and 2, then

$$\rho_1 A_1 v_1 = \rho_2 A_2 v_2 \qquad \text{(continuity equation, compressible fluid)} \tag{12.12}$$

If the fluid is denser at point 2 than at point 1 $(\rho_2 > \rho_1)$, the volume flow rate at point 2 will be less than at point 1 $(A_2 v_2 < A_1 v_1)$. We leave the details to you. If the fluid is incompressible so that ρ_1 and ρ_2 are always equal, Eq. (12.12) reduces to Eq. (12.10).

Example 12.6 Flow of an incompressible fluid

Incompressible oil of density 850 kg/m^3 is pumped through a cylindrical pipe at a rate of 9.5 liters per second. (a) The first section of the pipe has a diameter of 8.0 cm. What is the flow speed of the oil? What is the mass flow rate? (b) The second section of the pipe has a diameter of 4.0 cm. What are the flow speed and mass flow rate in that section?

SOLUTION

IDENTIFY and SET UP: Since the oil is incompressible, the volume flow rate has the *same* value (9.5 L/s) in both sections of pipe. The mass flow rate (the density times the volume flow rate) also has the same value in both sections. (This is just the statement that no fluid is lost or added anywhere along the pipe.) We use the volume flow rate equation, Eq. (12.11), to determine the speed v_1 in the 8.0-cm-diameter section and the continuity equation for incompressible flow, Eq. (12.10), to find the speed v_2 in the 4.0-cm-diameter section.

EXECUTE: (a) From Eq. (12.11) the volume flow rate in the first section is $dV/dt = A_1 v_1$, where A_1 is the cross-sectional area of the pipe of diameter 8.0 cm and radius 4.0 cm. Hence

$$v_1 = \frac{dV/dt}{A_1} = \frac{(9.5\ \text{L/s})(10^{-3}\ \text{m}^3/\text{L})}{\pi(4.0 \times 10^{-2}\ \text{m})^2} = 1.9\ \text{m/s}$$

The mass flow rate is $\rho\, dV/dt = (850\ \text{kg/m}^3)(9.5 \times 10^{-3}\ \text{m}^3/\text{s}) = 8.1\ \text{kg/s}$.

(b) From the continuity equation, Eq. (12.10),

$$v_2 = \frac{A_1}{A_2} v_1 = \frac{\pi(4.0 \times 10^{-2}\ \text{m})^2}{\pi(2.0 \times 10^{-2}\ \text{m})^2}(1.9\ \text{m/s}) = 7.6\ \text{m/s} = 4v_1$$

The volume and mass flow rates are the same as in part (a).

EVALUATE: The second section of pipe has one-half the diameter and one-fourth the cross-sectional area of the first section. Hence the speed must be four times greater in the second section, which is just what our result shows.

Test Your Understanding of Section 12.4 A maintenance crew is working on a section of a three-lane highway, leaving only one lane open to traffic. The result is much slower traffic flow (a traffic jam). Do cars on a highway behave like (i) the molecules of an incompressible fluid or (ii) the molecules of a compressible fluid?

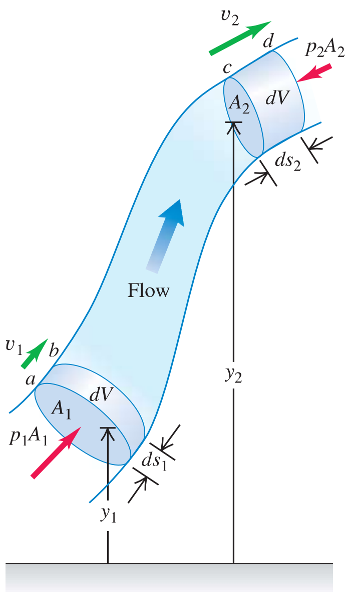
12.5 Bernoulli's Equation

According to the continuity equation, the speed of fluid flow can vary along the paths of the fluid. The pressure can also vary; it depends on height as in the static situation (see Section 12.2), and it also depends on the speed of flow. We can derive an important relationship called *Bernoulli's equation* that relates the pressure, flow speed, and height for flow of an ideal, incompressible fluid. Bernoulli's equation is an essential tool in analyzing plumbing systems, hydroelectric generating stations, and the flight of airplanes.

The dependence of pressure on speed follows from the continuity equation, Eq. (12.10). When an incompressible fluid flows along a flow tube with varying cross section, its speed *must* change, and so an element of fluid must have an acceleration. If the tube is horizontal, the force that causes this acceleration has to be applied by the surrounding fluid. This means that the pressure *must* be different in regions of different cross section; if it were the same everywhere, the net force on every fluid element would be zero. When a horizontal flow tube narrows and a fluid element speeds up, it must be moving toward a region of lower pressure in order to have a net forward force to accelerate it. If the elevation also changes, this causes an additional pressure difference.

Deriving Bernoulli's Equation

To derive Bernoulli's equation, we apply the work–energy theorem to the fluid in a section of a flow tube. In Fig. 12.22 we consider the element of fluid that at some initial time lies between the two cross sections a and c. The speeds at the lower and upper ends are v_1 and v_2. In a small time interval dt, the fluid that is initially at a moves to b, a distance $ds_1 = v_1\,dt$, and the fluid that is initially at c moves to d, a distance $ds_2 = v_2\,dt$. The cross-sectional areas at the two ends are A_1 and A_2, as shown. The fluid is incompressible; hence by the continuity equation, Eq. (12.10), the volume of fluid dV passing *any* cross section during time dt is the same. That is, $dV = A_1\,ds_1 = A_2\,ds_2$.



12.22 Deriving Bernoulli's equation. The net work done on a fluid element by the pressure of the surrounding fluid equals the change in the kinetic energy plus the change in the gravitational potential energy.

Let's compute the *work* done on this fluid element during dt. We assume that there is negligible internal friction in the fluid (i.e., no viscosity), so the only nongravitational forces that do work on the fluid element are due to the pressure of the surrounding fluid. The pressures at the two ends are p_1 and p_2; the force on the cross section at a is p_1A_1, and the force at c is p_2A_2. The net work dW done on the element by the surrounding fluid during this displacement is therefore

$$dW = p_1A_1\,ds_1 - p_2A_2\,ds_2 = (p_1 - p_2)dV \qquad (12.13)$$

The second term has a negative sign because the force at c opposes the displacement of the fluid.

The work dW is due to forces other than the conservative force of gravity, so it equals the change in the total mechanical energy (kinetic energy plus gravitational potential energy) associated with the fluid element. The mechanical energy for the fluid between sections b and c does not change. At the beginning of dt the fluid between a and b has volume $A_1\,ds_1$, mass $\rho A_1\,ds_1$, and kinetic energy $\frac{1}{2}\rho(A_1\,ds_1)v_1^2$. At the end of dt the fluid between c and d has kinetic energy $\frac{1}{2}\rho(A_2\,ds_2)v_2^2$. The net change in kinetic energy dK during time dt is

$$dK = \tfrac{1}{2}\rho\,dV(v_2^2 - v_1^2) \qquad (12.14)$$

What about the change in gravitational potential energy? At the beginning of dt, the potential energy for the mass between a and b is $dm\,gy_1 = \rho\,dV\,gy_1$. At

the end of dt, the potential energy for the mass between c and d is $dm\,gy_2 = \rho\,dV\,gy_2$. The net change in potential energy dU during dt is

$$dU = \rho\,dV\,g(y_2 - y_1) \tag{12.15}$$

Combining Eqs. (12.13), (12.14), and (12.15) in the energy equation $dW = dK + dU$, we obtain

$$\begin{aligned}(p_1 - p_2)dV &= \tfrac{1}{2}\rho\,dV(v_2^2 - v_1^2) + \rho\,dV\,g(y_2 - y_1)\\ p_1 - p_2 &= \tfrac{1}{2}\rho(v_2^2 - v_1^2) + \rho g(y_2 - y_1)\end{aligned} \tag{12.16}$$

This is **Bernoulli's equation.** It states that the work done on a unit volume of fluid by the surrounding fluid is equal to the sum of the changes in kinetic and potential energies per unit volume that occur during the flow. We may also interpret Eq. (12.16) in terms of pressures. The first term on the right is the pressure difference associated with the change of speed of the fluid. The second term on the right is the additional pressure difference caused by the weight of the fluid and the difference in elevation of the two ends.

We can also express Eq. (12.16) in a more convenient form as

$$p_1 + \rho g y_1 + \tfrac{1}{2}\rho v_1^2 = p_2 + \rho g y_2 + \tfrac{1}{2}\rho v_2^2 \quad \text{(Bernoulli's equation)} \tag{12.17}$$

The subscripts 1 and 2 refer to *any* two points along the flow tube, so we can also write

$$p + \rho g y + \tfrac{1}{2}\rho v^2 = \text{constant} \tag{12.18}$$

Note that when the fluid is *not* moving (so $v_1 = v_2 = 0$), Eq. (12.17) reduces to the pressure relationship we derived for a fluid at rest, Eq. (12.5).

CAUTION **Bernoulli's principle applies only in certain situations** We stress again that Bernoulli's equation is valid for only incompressible, steady flow of a fluid with no internal friction (no viscosity). It's a simple equation that's easy to use; don't let this tempt you to use it in situations in which it doesn't apply!

Problem-Solving Strategy 12.1 Bernoulli's Equation

Bernoulli's equation is derived from the work–energy theorem, so much of Problem-Solving Strategy 7.1 (Section 7.1) is applicable here.

IDENTIFY *the relevant concepts:* Bernoulli's equation is applicable to steady flow of an incompressible fluid that has no internal friction (see Section 12.6). It is generally applicable to flows through large pipes and to flows within bulk fluids (e.g., air flowing around an airplane or water flowing around a fish).

SET UP *the problem* using the following steps:

1. Identify the points 1 and 2 referred to in Bernoulli's equation, Eq. (12.17).
2. Define your coordinate system, particularly the level at which $y = 0$. Take the positive y-direction to be upward.
3. Make lists of the unknown and known quantities in Eq. (12.17). Decide which unknowns are the target variables.

EXECUTE *the solution* as follows: Write Bernoulli's equation and solve for the unknowns. You may need the continuity equation, Eq. (12.10), to get a relationship between the two speeds in terms of cross-sectional areas of pipes or containers. You may also need Eq. (12.11) to find the volume flow rate.

EVALUATE *your answer:* Verify that the results make physical sense. Check that you have used consistent units: In SI units, pressure is in pascals, density in kilograms per cubic meter, and speed in meters per second. Also note that the pressures must be either *all* absolute pressures or *all* gauge pressures.

Example 12.7 Water pressure in the home

Water enters a house (Fig. 12.23) through a pipe with an inside diameter of 2.0 cm at an absolute pressure of 4.0×10^5 Pa (about 4 atm). A 1.0-cm-diameter pipe leads to the second-floor bathroom 5.0 m above. When the flow speed at the inlet pipe is 1.5 m/s, find the flow speed, pressure, and volume flow rate in the bathroom.

SOLUTION

IDENTIFY and SET UP: We assume that the water flows at a steady rate. Water is effectively incompressible, so we can use the continuity equation. It's reasonable to ignore internal friction because the pipe has a relatively large diameter, so we can also use Bernoulli's equation. Let points 1 and 2 be at the inlet pipe and at the bathroom, respectively. We are given the pipe diameters at points 1 and 2, from which we calculate the areas A_1 and A_2, as well as the speed $v_1 = 1.5$ m/s and pressure $p_1 = 4.0 \times 10^5$ Pa at the inlet pipe. We take $y_1 = 0$ and $y_2 = 5.0$ m. We find the speed v_2 using the continuity equation and the pressure p_2 using Bernoulli's equation. Knowing v_2, we calculate the volume flow rate $v_2 A_2$.

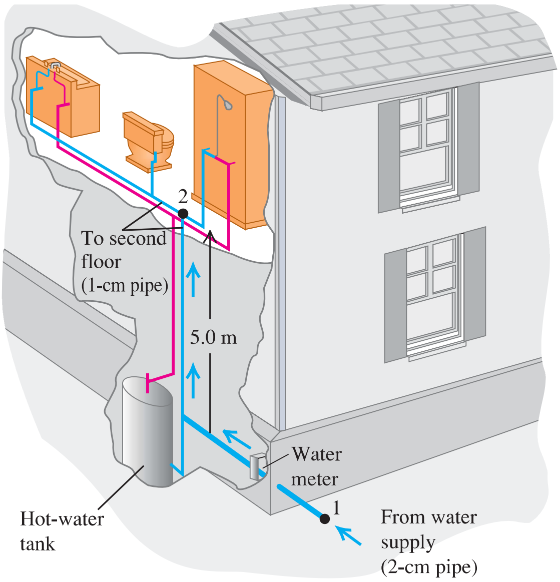
EXECUTE: From the continuity equation, Eq. (12.10),

$$v_2 = \frac{A_1}{A_2}v_1 = \frac{\pi(1.0\ \text{cm})^2}{\pi(0.50\ \text{cm})^2}(1.5\ \text{m/s}) = 6.0\ \text{m/s}$$

From Bernoulli's equation, Eq. (12.16),

$$\begin{aligned}p_2 &= p_1 - \tfrac{1}{2}\rho(v_2^2 - v_1^2) - \rho g(y_2 - y_1)\\ &= 4.0 \times 10^5\ \text{Pa}\\ &\quad - \tfrac{1}{2}(1.0 \times 10^3\ \text{kg/m}^3)(36\ \text{m}^2/\text{s}^2 - 2.25\ \text{m}^2/\text{s}^2)\\ &\quad - (1.0 \times 10^3\ \text{kg/m}^3)(9.8\ \text{m/s}^2)(5.0\ \text{m})\\ &= 4.0 \times 10^5\ \text{Pa} - 0.17 \times 10^5\ \text{Pa} - 0.49 \times 10^5\ \text{Pa}\\ &= 3.3 \times 10^5\ \text{Pa} = 3.3\ \text{atm} = 48\ \text{lb/in.}^2\end{aligned}$$

12.23 What is the water pressure in the second-story bathroom of this house?



The volume flow rate is

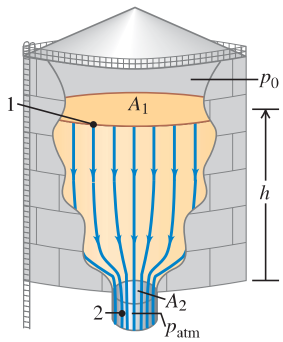
$$\begin{aligned}\frac{dV}{dt} &= A_2 v_2 = \pi(0.50 \times 10^{-2}\ \text{m})^2(6.0\ \text{m/s})\\ &= 4.7 \times 10^{-4}\ \text{m}^3/\text{s} = 0.47\ \text{L/s}\end{aligned}$$

EVALUATE: This is a reasonable flow rate for a bathroom faucet or shower. Note that if the water is turned off, v_1 and v_2 are both zero, the term $\frac{1}{2}\rho(v_2^2 - v_1^2)$ in Bernoulli's equation vanishes, and p_2 rises from 3.3×10^5 Pa to 3.5×10^5 Pa.

Example 12.8 Speed of efflux

Figure 12.24 shows a gasoline storage tank with cross-sectional area A_1, filled to a depth h. The space above the gasoline contains air at pressure p_0, and the gasoline flows out the bottom of the tank through a short pipe with cross-sectional area A_2. Derive expressions for the flow speed in the pipe and the volume flow rate.

12.24 Calculating the speed of efflux for gasoline flowing out the bottom of a storage tank.



SOLUTION

IDENTIFY and SET UP: We consider the entire volume of moving liquid as a single flow tube of an incompressible fluid with negligible internal friction. Hence, we can use Bernoulli's equation. Points 1 and 2 are at the surface of the gasoline and at the exit pipe, respectively. At point 1 the pressure is p_0, which we assume to be fixed; at point 2 it is atmospheric pressure p_{atm}. We take $y = 0$ at the exit pipe, so $y_1 = h$ and $y_2 = 0$. Because A_1 is very much larger than A_2, the upper surface of the gasoline will drop very slowly and we can regard v_1 as essentially equal to zero. We find v_2 from Eq. (12.17) and the volume flow rate from Eq. (12.11).

EXECUTE: We apply Bernoulli's equation to points 1 and 2:

$$p_0 + \tfrac{1}{2}\rho v_1^2 + \rho g h = p_{atm} + \tfrac{1}{2}\rho v_2^2 + \rho g(0)$$

$$v_2^2 = v_1^2 + 2\left(\frac{p_0 - p_{atm}}{\rho}\right) + 2gh$$

Continued

Using $v_1 = 0$, we find

$$v_2 = \sqrt{2\left(\frac{p_0 - p_{atm}}{\rho}\right) + 2gh}$$

From Eq. (12.11), the volume flow rate is $dV/dt = v_2A_2$.

EVALUATE: The speed v_2, sometimes called the *speed of efflux,* depends on both the pressure difference $(p_0 - p_{atm})$ and the height h of the liquid level in the tank. If the top of the tank is vented to the atmosphere, $p_0 = p_{atm}$ and $p_0 - p_{atm} = 0$. Then

$$v_2 = \sqrt{2gh}$$

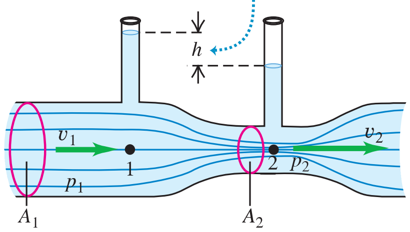
That is, the speed of efflux from an opening at a distance h below the top surface of the liquid is the *same* as the speed a body would acquire in falling freely through a height h. This result is called *Torricelli's theorem.* It is valid not only for an opening in the bottom of a container, but also for a hole in a side wall at a depth h below the surface. In this case the volume flow rate is

$$\frac{dV}{dt} = A_2\sqrt{2gh}$$

Example 12.9 The Venturi meter

Figure 12.25 shows a *Venturi meter,* used to measure flow speed in a pipe. Derive an expression for the flow speed v_1 in terms of the cross-sectional areas A_1 and A_2 and the difference in height h of the liquid levels in the two vertical tubes.

12.25 The Venturi meter.



SOLUTION

IDENTIFY and SET UP: The flow is steady, and we assume the fluid is incompressible and has negligible internal friction. Hence we can use Bernoulli's equation. We apply that equation to the wide part (point 1) and narrow part (point 2, the *throat*) of the pipe. Equation (12.6) relates h to the pressure difference $p_1 - p_2$.

EXECUTE: Points 1 and 2 have the same vertical coordinate $y_1 = y_2$, so Eq. (12.17) says

$$p_1 + \tfrac{1}{2}\rho v_1^2 = p_2 + \tfrac{1}{2}\rho v_2^2$$

From the continuity equation, $v_2 = (A_1/A_2)v_1$. Substituting this and rearranging, we get

$$p_1 - p_2 = \tfrac{1}{2}\rho v_1^2\left[\left(\frac{A_1}{A_2}\right)^2 - 1\right]$$

From Eq. (12.6), the pressure difference $p_1 - p_2$ is also equal to ρgh. Substituting this and solving for v_1, we get

$$v_1 = \sqrt{\frac{2gh}{(A_1/A_2)^2 - 1}}$$

EVALUATE: Because A_1 is greater than A_2, v_2 is greater than v_1 and the pressure p_2 in the throat is *less* than p_1. Those pressure differences produce a net force to the right that makes the fluid speed up as it enters the throat, and a net force to the left that slows it as it leaves.

Conceptual Example 12.10 Lift on an airplane wing

Figure 12.26a shows flow lines around a cross section of an airplane wing. The flow lines crowd together above the wing, corresponding to increased flow speed and reduced pressure, just as in the Venturi throat in Example 12.9. Hence the downward force of the air on the top side of the wing is less than the upward force of the air on the underside of the wing, and there is a net upward force or *lift.* Lift is not simply due to the impulse of air striking the underside of the wing; in fact, the reduced pressure on the upper wing surface makes the greatest contribution to the lift. (This simplified discussion ignores the formation of vortices.)

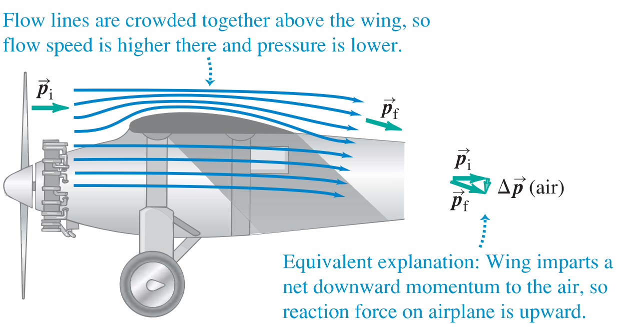
We can also understand the lift force on the basis of momentum changes. The vector diagram in Fig. 12.26a shows that there is a net *downward* change in the vertical component of momentum of the air flowing past the wing, corresponding to the downward force the wing exerts on the air. The reaction force *on* the wing is *upward,* as we concluded above.

Similar flow patterns and lift forces are found in the vicinity of any humped object in a wind. A moderate wind makes an umbrella "float"; a strong wind can turn it inside out. At high speed, lift can reduce traction on a car's tires; a "spoiler" at the car's tail, shaped like an upside-down wing, provides a compensating downward force.

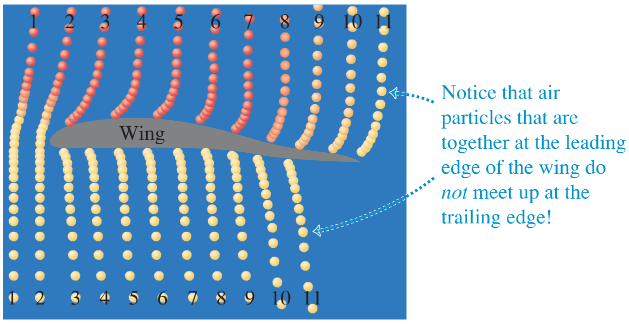
CAUTION **A misconception about wings** Some discussions of lift claim that air travels faster over the top of a wing because "it has farther to travel." This claim assumes that air molecules that part company at the front of the wing, one traveling over the wing and one under it, must meet again at the wing's trailing edge. Not so! Figure 12.26b shows a computer simulation of parcels of air flowing around an airplane wing. Parcels that are adjacent at the front of the wing do *not* meet at the trailing edge; the flow over the top of the wing is much faster than if the parcels had to meet. In accordance with Bernoulli's equation, this faster speed means that there is even lower pressure above the wing (and hence greater lift) than the "farther-to-travel" claim would suggest.

12.26 Flow around an airplane wing.

(a) Flow lines around an airplane wing



(b) Computer simulation of air parcels flowing around a wing, showing that air moves much faster over the top than over the bottom.



Test Your Understanding of Section 12.5 Which is the most accurate statement of Bernoulli's principle? (i) Fast-moving air causes lower pressure; (ii) lower pressure causes fast-moving air; (iii) both (i) and (ii) are equally accurate.

12.6 Viscosity and Turbulence

In our discussion of fluid flow we assumed that the fluid had no internal friction and that the flow was laminar. While these assumptions are often quite valid, in many important physical situations the effects of viscosity (internal friction) and turbulence (nonlaminar flow) are extremely important. Let's take a brief look at some of these situations.

12.27 Lava is an example of a viscous fluid. The viscosity decreases with increasing temperature: The hotter the lava, the more easily it can flow.

Viscosity

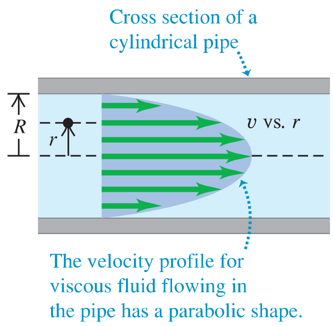
Viscosity is internal friction in a fluid. Viscous forces oppose the motion of one portion of a fluid relative to another. Viscosity is the reason it takes effort to paddle a canoe through calm water, but it is also the reason the paddle works. Viscous effects are important in the flow of fluids in pipes, the flow of blood, the lubrication of engine parts, and many other situations.

Fluids that flow readily, such as water or gasoline, have smaller viscosities than do "thick" liquids such as honey or motor oil. Viscosities of all fluids are strongly temperature dependent, increasing for gases and decreasing for liquids as the temperature increases (Fig. 12.27). Oils for engine lubrication must flow equally well in cold and warm conditions, and so are designed to have as *little* temperature variation of viscosity as possible.

A viscous fluid always tends to cling to a solid surface in contact with it. There is always a thin *boundary layer* of fluid near the surface, in which the fluid is nearly at rest with respect to the surface. That's why dust particles can cling to a fan blade even when it is rotating rapidly, and why you can't get all the dirt off your car by just squirting a hose at it.

Viscosity has important effects on the flow of liquids through pipes, including the flow of blood in the circulatory system. First think about a fluid with zero viscosity so that we can apply Bernoulli's equation, Eq. (12.17). If the two ends of a long cylindrical pipe are at the same height $(y_1 = y_2)$ and the flow speed is the same at both ends (so $v_1 = v_2$), Bernoulli's equation tells us that the pressure is the same at both ends of the pipe. But this result simply isn't true if we take viscosity into account. To see why, consider Fig. 12.28, which shows the flow-speed profile for laminar flow of a viscous fluid in a long cylindrical pipe. Due to viscosity, the speed is *zero* at the pipe walls (to which the fluid clings) and is greatest at the center of the pipe. The motion is like a lot of concentric tubes sliding relative to

12.28 Velocity profile for a viscous fluid in a cylindrical pipe.



Application Listening for Turbulent Flow
Normal blood flow in the human aorta is laminar, but a small disturbance such as a heart pathology can cause the flow to become turbulent. Turbulence makes noise, which is why listening to blood flow with a stethoscope is a useful diagnostic technique.

one another, with the central tube moving fastest and the outermost tube at rest. Viscous forces between the tubes oppose this sliding, so to keep the flow going we must apply a greater pressure at the back of the flow than at the front. That's why you have to keep squeezing a tube of toothpaste or a packet of ketchup (both viscous fluids) to keep the fluid coming out of its container. Your fingers provide a pressure at the back of the flow that is far greater than the atmospheric pressure at the front of the flow.

The pressure difference required to sustain a given volume flow rate through a cylindrical pipe of length L and radius R turns out to be proportional to L/R^4. If we decrease R by one-half, the required pressure increases by $2^4 = 16$; decreasing R by a factor of 0.90 (a 10% reduction) increases the required pressure difference by a factor of $(1/0.90)^4 = 1.52$ (a 52% increase). This simple relationship explains the connection between a high-cholesterol diet (which tends to narrow the arteries) and high blood pressure. Due to the R^4 dependence, even a small narrowing of the arteries can result in substantially elevated blood pressure and added strain on the heart muscle.

Turbulence

When the speed of a flowing fluid exceeds a certain critical value, the flow is no longer laminar. Instead, the flow pattern becomes extremely irregular and complex, and it changes continuously with time; there is no steady-state pattern. This irregular, chaotic flow is called **turbulence.** Figure 12.20 shows the contrast between laminar and turbulent flow for smoke rising in air. Bernoulli's equation is *not* applicable to regions where there is turbulence because the flow is not steady.

Whether a flow is laminar or turbulent depends in part on the fluid's viscosity. The greater the viscosity, the greater the tendency for the fluid to flow in sheets or lamina and the more likely the flow is to be laminar. (When we discussed Bernoulli's equation in Section 12.5, we assumed that the flow was laminar and that the fluid had zero viscosity. In fact, a *little* viscosity is needed to ensure that the flow is laminar.)

For a fluid of a given viscosity, flow speed is a determining factor for the onset of turbulence. A flow pattern that is stable at low speeds suddenly becomes unstable when a critical speed is reached. Irregularities in the flow pattern can be caused by roughness in the pipe wall, variations in the density of the fluid, and many other factors. At low flow speeds, these disturbances damp out; the flow pattern is *stable* and tends to maintain its laminar nature (Fig. 12.29a). When the critical speed is reached, however, the flow pattern becomes unstable. The disturbances no longer damp out but grow until they destroy the entire laminar-flow pattern (Fig. 12.29b).

(a) (b)

12.29 The flow of water from a faucet is (a) laminar at low speeds but (b) turbulent at sufficiently high speeds.

Conceptual Example 12.11 The curve ball

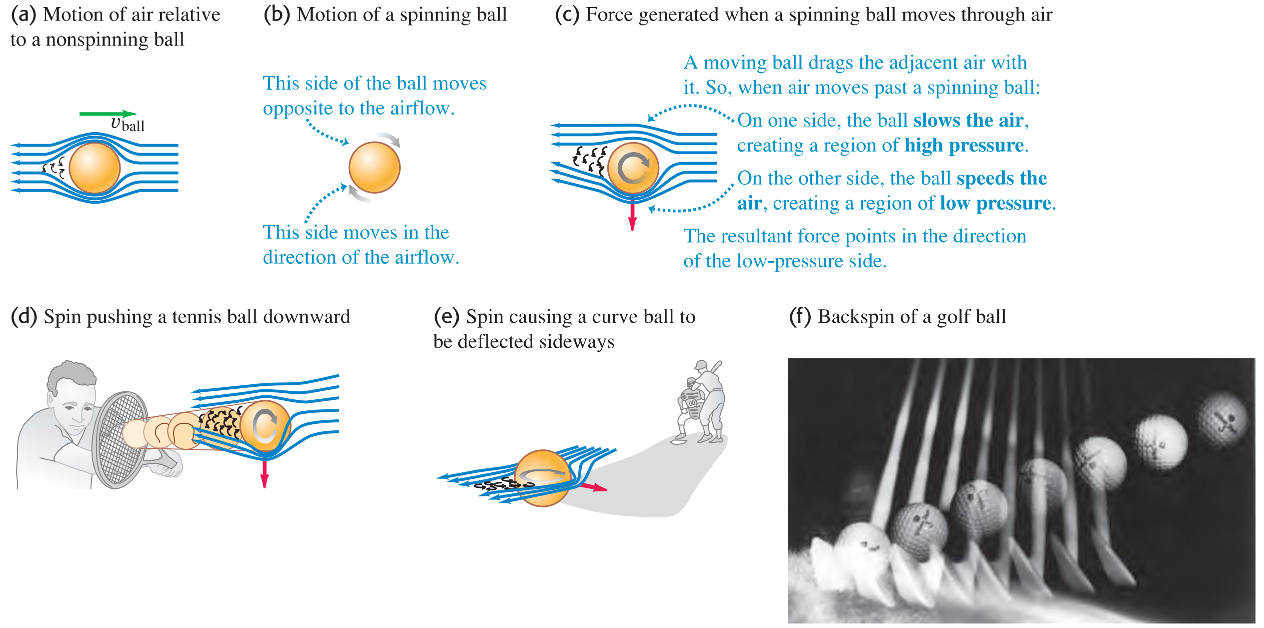
Does a curve ball *really* curve? Yes, it certainly does, and the reason is turbulence. Figure 12.30a shows a nonspinning ball moving through the air from left to right. The flow lines show that to an observer moving with the ball, the air stream appears to move from right to left. Because of the high speeds that are ordinarily involved (near 35 m/s, or 75 mi/h), there is a region of *turbulent* flow behind the ball.

Figure 12.30b shows a *spinning* ball with "top spin." Layers of air near the ball's surface are pulled around in the direction of the spin by friction between the ball and air and by the air's internal friction (viscosity). Hence air moves relative to the ball's surface more slowly at the top of the ball than at the bottom, and turbulence occurs farther forward on the top side than on the bottom. This asymmetry causes a pressure difference; the average pressure at the top of the ball is now greater than that at the bottom. As Fig. 12.30c shows, the resulting net force deflects the ball downward. "Top spin" is used in tennis to keep a fast serve in the court (Fig. 12.30d).

In baseball, a curve ball spins about a nearly *vertical* axis and the resulting deflection is sideways. In that case, Fig. 12.30c is a *top* view of the situation. A curve ball thrown by a left-handed pitcher spins as shown in Fig. 12.30e and will curve *toward* a right-handed batter, making it harder to hit.

A similar effect occurs with golf balls, which acquire "back spin" from impact with the grooved, slanted club face. Figure 12.30f shows the backspin of a golf ball just after impact. The resulting pressure difference between the top and bottom of the ball causes a *lift* force that keeps the ball in the air longer than would be possible without spin. A well-hit drive appears, from the tee, to "float" or even curve *upward* during the initial portion of its flight. This is a real effect, not an illusion. The dimples on the golf ball play an essential role; the viscosity of air gives a dimpled ball a much longer trajectory than an undimpled one with the same initial velocity and spin.

12.30 (a)–(e) Analyzing the motion of a spinning ball through the air. (f) Stroboscopic photograph of a golf ball being struck by a club. The picture was taken at 1000 flashes per second. The ball rotates about once in eight pictures, corresponding to an angular speed of 125 rev/s, or 7500 rpm.



Test Your Understanding of Section 12.6 How much more thumb pressure must a nurse use to administer an injection with a hypodermic needle of inside diameter 0.30 mm compared to one with inside diameter 0.60 mm? Assume that the two needles have the same length and that the volume flow rate is the same in both cases. (i) twice as much; (ii) 4 times as much; (iii) 8 times as much; (iv) 16 times as much; (v) 32 times as much.

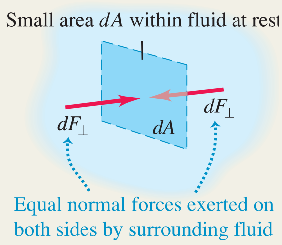
CHAPTER 12 SUMMARY

Density and pressure: Density is mass per unit volume. If a mass m of homogeneous material has volume V, its density ρ is the ratio m/V. Specific gravity is the ratio of the density of a material to the density of water. (See Example 12.1.)

Pressure is normal force per unit area. Pascal's law states that pressure applied to an enclosed fluid is transmitted undiminished to every portion of the fluid. Absolute pressure is the total pressure in a fluid; gauge pressure is the difference between absolute pressure and atmospheric pressure. The SI unit of pressure is the pascal (Pa): 1 Pa $=$ 1 N/m^2. (See Example 12.2.)

$$\rho = \frac{m}{V} \quad (12.1)$$

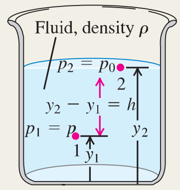
$$p = \frac{dF_\perp}{dA} \quad (12.2)$$



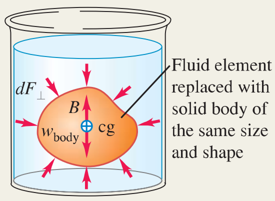
Pressures in a fluid at rest: The pressure difference between points 1 and 2 in a static fluid of uniform density ρ (an incompressible fluid) is proportional to the difference between the elevations y_1 and y_2. If the pressure at the surface of an incompressible liquid at rest is p_0, then the pressure at a depth h is greater by an amount $\rho g h$. (See Examples 12.3 and 12.4.)

$$p_2 - p_1 = -\rho g(y_2 - y_1) \quad (12.5)$$
(pressure in a fluid of uniform density)

$$p = p_0 + \rho g h \quad (12.6)$$
(pressure in a fluid of uniform density)



Buoyancy: Archimedes's principle states that when a body is immersed in a fluid, the fluid exerts an upward buoyant force on the body equal to the weight of the fluid that the body displaces. (See Example 12.5.)



Fluid flow: An ideal fluid is incompressible and has no viscosity (no internal friction). A flow line is the path of a fluid particle; a streamline is a curve tangent at each point to the velocity vector at that point. A flow tube is a tube bounded at its sides by flow lines. In laminar flow, layers of fluid slide smoothly past each other. In turbulent flow, there is great disorder and a constantly changing flow pattern.

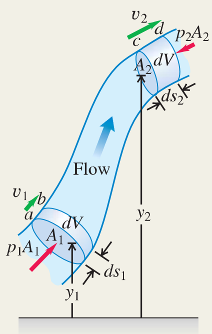
Conservation of mass in an incompressible fluid is expressed by the continuity equation, which relates the flow speeds v_1 and v_2 for two cross sections A_1 and A_2 in a flow tube. The product Av equals the volume flow rate, dV/dt, the rate at which volume crosses a section of the tube. (See Example 12.6.)

Bernoulli's equation relates the pressure p, flow speed v, and elevation y for any two points, assuming steady flow in an ideal fluid. (See Examples 12.7–12.10.)

$$A_1 v_1 = A_2 v_2 \quad (12.10)$$
(continuity equation, incompressible fluid)

$$\frac{dV}{dt} = Av \quad (12.11)$$
(volume flow rate)

$$p_1 + \rho g y_1 + \tfrac{1}{2}\rho v_1^2 = p_2 + \rho g y_2 + \tfrac{1}{2}\rho v_2^2 \quad (12.17)$$
(Bernoulli's equation)



BRIDGING PROBLEM How Long to Drain?

A large cylindrical tank with diameter D is open to the air at the top. The tank contains water to a height H. A small circular hole with diameter d, where d is very much less than D, is then opened at the bottom of the tank. Ignore any effects of viscosity. (a) Find y, the height of water in the tank a time t after the hole is opened, as a function of t. (b) How long does it take to drain the tank completely? (c) If you double the initial height of water in the tank, by what factor does the time to drain the tank increase?

SOLUTION GUIDE

See MasteringPhysics® study area for a Video Tutor solution.

IDENTIFY and SET UP

1. Draw a sketch of the situation that shows all of the relevant dimensions.
2. Make a list of the unknown quantities, and decide which of these are the target variables.
3. What is the speed at which water flows out of the bottom of the tank? How is this related to the volume flow rate of water out of the tank? How is the volume flow rate related to the rate of change of y?

EXECUTE

4. Use your results from step 3 to write an equation for dy/dt.
5. Your result from step 4 is a relatively simple differential equation. With your knowledge of calculus, you can integrate it to find y as a function of t. (*Hint:* Once you've done the integration, you'll still have to do a little algebra.)
6. Use your result from step 5 to find the time when the tank is empty. How does your result depend on the initial height H?

EVALUATE

7. Check whether your answers are reasonable. A good check is to draw a graph of y versus t. According to your graph, what is the algebraic sign of dy/dt at different times? Does this make sense?

Problems

For instructor-assigned homework, go to www.masteringphysics.com

•, ••, •••: Problems of increasing difficulty. **CP**: Cumulative problems incorporating material from earlier chapters. **CALC**: Problems requiring calculus. **BIO**: Biosciences problems.

DISCUSSION QUESTIONS

Q12.1 A cube of oak wood with very smooth faces normally floats in water. Suppose you submerge it completely and press one face flat against the bottom of a tank so that no water is under that face. Will the block float to the surface? Is there a buoyant force on it? Explain.

Q12.2 A rubber hose is attached to a funnel, and the free end is bent around to point upward. When water is poured into the funnel, it rises in the hose to the same level as in the funnel, even though the funnel has a lot more water in it than the hose does. Why? What supports the extra weight of the water in the funnel?

Q12.3 Comparing Example 12.1 (Section 12.1) and Example 12.2 (Section 12.2), it seems that 700 N of air is exerting a downward force of 2.0×10^6 N on the floor. How is this possible?

Q12.4 Equation (12.7) shows that an area ratio of 100 to 1 can give 100 times more output force than input force. Doesn't this violate conservation of energy? Explain.

Q12.5 You have probably noticed that the lower the tire pressure, the larger the contact area between the tire and the road. Why?

Q12.6 In hot-air ballooning, a large balloon is filled with air heated by a gas burner at the bottom. Why must the air be heated? How does the balloonist control ascent and descent?

Q12.7 In describing the size of a large ship, one uses such expressions as "it displaces 20,000 tons." What does this mean? Can the weight of the ship be obtained from this information?

Q12.8 You drop a solid sphere of aluminum in a bucket of water that sits on the ground. The buoyant force equals the weight of water displaced; this is less than the weight of the sphere, so the sphere sinks to the bottom. If you take the bucket with you on an elevator that accelerates upward, the apparent weight of the water increases and the buoyant force on the sphere increases. Could the acceleration of the elevator be great enough to make the sphere pop up out of the water? Explain.

Q12.9 A rigid, lighter-than-air dirigible filled with helium cannot continue to rise indefinitely. Why? What determines the maximum height it can attain?

Q12.10 Air pressure decreases with increasing altitude. So why is air near the surface not continuously drawn upward toward the lower-pressure regions above?

Q12.11 The purity of gold can be tested by weighing it in air and in water. How? Do you think you could get away with making a fake gold brick by gold-plating some cheaper material?

Q12.12 During the Great Mississippi Flood of 1993, the levees in St. Louis tended to rupture first at the bottom. Why?

Q12.13 A cargo ship travels from the Atlantic Ocean (salt water) to Lake Ontario (freshwater) via the St. Lawrence River. The ship rides several centimeters lower in the water in Lake Ontario than it did in the ocean. Explain why.

Q12.14 You push a piece of wood under the surface of a swimming pool. After it is completely submerged, you keep pushing it deeper and deeper. As you do this, what will happen to the buoyant force on it? Will the force keep increasing, stay the same, or decrease? Why?

Q12.15 An old question is "Which weighs more, a pound of feathers or a pound of lead?" If the weight in pounds is the gravitational force, will a pound of feathers balance a pound of lead on opposite pans of an equal-arm balance? Explain, taking into account buoyant forces.

Q12.16 Suppose the door of a room makes an airtight but frictionless fit in its frame. Do you think you could open the door if the air pressure on one side were standard atmospheric pressure and the air pressure on the other side differed from standard by 1%? Explain.

Q12.17 At a certain depth in an incompressible liquid, the absolute pressure is p. At twice this depth, will the absolute pressure be equal to $2p$, greater than $2p$, or less than $2p$? Justify your answer.

Q12.18 A piece of iron is glued to the top of a block of wood. When the block is placed in a bucket of water with the iron on top, the block floats. The block is now turned over so that the iron is submerged beneath the wood. Does the block float or sink? Does the water level in the bucket rise, drop, or stay the same? Explain your answers.

Q12.19 You take an empty glass jar and push it into a tank of water with the open mouth of the jar downward, so that the air inside the jar is trapped and cannot get out. If you push the jar deeper into the water, does the buoyant force on the jar stay the same? If not, does it increase or decrease? Explain your answer.

Q12.20 You are floating in a canoe in the middle of a swimming pool. Your friend is at the edge of the pool, carefully noting the level of the water on the side of the pool. You have a bowling ball with you in the canoe. If you carefully drop the bowling ball over the side of the canoe and it sinks to the bottom of the pool, does the water level in the pool rise or fall?

Q12.21 You are floating in a canoe in the middle of a swimming pool. A large bird flies up and lights on your shoulder. Does the water level in the pool rise or fall?

Q12.22 At a certain depth in the incompressible ocean the gauge pressure is p_g. At three times this depth, will the gauge pressure be greater than $3p_g$, equal to $3p_g$, or less than $3p_g$? Justify your answer.

Q12.23 An ice cube floats in a glass of water. When the ice melts, will the water level in the glass rise, fall, or remain unchanged? Explain.

Q12.24 You are told, "Bernoulli's equation tells us that where there is higher fluid speed, there is lower fluid pressure, and vice versa." Is this statement always true, even for an idealized fluid? Explain.

Q12.25 If the velocity at each point in space in steady-state fluid flow is constant, how can a fluid particle accelerate?

Q12.26 In a store-window vacuum cleaner display, a table-tennis ball is suspended in midair in a jet of air blown from the outlet hose of a tank-type vacuum cleaner. The ball bounces around a little but always moves back toward the center of the jet, even if the jet is tilted from the vertical. How does this behavior illustrate Bernoulli's equation?

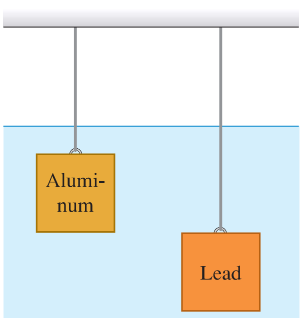
Q12.27 A tornado consists of a rapidly whirling air vortex. Why is the pressure always much lower in the center than at the outside? How does this condition account for the destructive power of a tornado?

Q12.28 Airports at high elevations have longer runways for takeoffs and landings than do airports at sea level. One reason is that aircraft engines develop less power in the thin air well above sea level. What is another reason?

Q12.29 When a smooth-flowing stream of water comes out of a faucet, it narrows as it falls. Explain why this happens.

Q12.30 Identical-size lead and aluminum cubes are suspended at different depths by two wires in a large vat of water (Fig. Q12.30). (a) Which cube experiences a greater buoyant force? (b) For which cube is the tension in the wire greater? (c) Which cube experiences a greater force on its lower face? (d) For which cube is the difference in pressure between the upper and lower faces greater?

Figure **Q12.30**



EXERCISES

Section 12.1 Density

12.1 •• On a part-time job, you are asked to bring a cylindrical iron rod of length 85.8 cm and diameter 2.85 cm from a storage room to a machinist. Will you need a cart? (To answer, calculate the weight of the rod.)

12.2 •• A cube 5.0 cm on each side is made of a metal alloy. After you drill a cylindrical hole 2.0 cm in diameter all the way through and perpendicular to one face, you find that the cube weighs 7.50 N. (a) What is the density of this metal? (b) What did the cube weigh before you drilled the hole in it?

12.3 • You purchase a rectangular piece of metal that has dimensions $5.0 \times 15.0 \times 30.0$ mm and mass 0.0158 kg. The seller tells you that the metal is gold. To check this, you compute the average density of the piece. What value do you get? Were you cheated?

12.4 •• **Gold Brick.** You win the lottery and decide to impress your friends by exhibiting a million-dollar cube of gold. At the time, gold is selling for \$426.60 per troy ounce, and 1.0000 troy ounce equals 31.1035 g. How tall would your million-dollar cube be?

12.5 •• A uniform lead sphere and a uniform aluminum sphere have the same mass. What is the ratio of the radius of the aluminum sphere to the radius of the lead sphere?

12.6 • (a) What is the average density of the sun? (b) What is the average density of a neutron star that has the same mass as the sun but a radius of only 20.0 km?

12.7 •• A hollow cylindrical copper pipe is 1.50 m long and has an outside diameter of 3.50 cm and an inside diameter of 2.50 cm. How much does it weigh?

Section 12.2 Pressure in a Fluid

12.8 •• **Black Smokers.** Black smokers are hot volcanic vents that emit smoke deep in the ocean floor. Many of them teem with exotic creatures, and some biologists think that life on earth may have begun around such vents. The vents range in depth from about 1500 m to 3200 m below the surface. What is the gauge pressure at a 3200-m deep vent, assuming that the density of water does not vary? Express your answer in pascals and atmospheres.

12.9 •• **Oceans on Mars.** Scientists have found evidence that Mars may once have had an ocean 0.500 km deep. The acceleration due to gravity on Mars is 3.71 m/s^2. (a) What would be the gauge pressure at the bottom of such an ocean, assuming it was freshwater? (b) To what depth would you need to go in the earth's ocean to experience the same gauge pressure?

12.10 •• **BIO** (a) Calculate the difference in blood pressure between the feet and top of the head for a person who is 1.65 m tall. (b) Consider a cylindrical segment of a blood vessel 2.00 cm long and 1.50 mm in diameter. What *additional* outward force would such a vessel need to withstand in the person's feet compared to a similar vessel in her head?

12.11 • **BIO** In intravenous feeding, a needle is inserted in a vein in the patient's arm and a tube leads from the needle to a reservoir of fluid (density 1050 kg/m^3) located at height h above the arm. The top of the reservoir is open to the air. If the gauge pressure inside the vein is 5980 Pa, what is the minimum value of h that allows fluid to enter the vein? Assume the needle diameter is large enough that you can ignore the viscosity (see Section 12.6) of the fluid.

12.12 • A barrel contains a 0.120-m layer of oil floating on water that is 0.250 m deep. The density of the oil is 600 kg/m^3. (a) What is the gauge pressure at the oil–water interface? (b) What is the gauge pressure at the bottom of the barrel?

12.13 • **BIO** **Standing on Your Head.** (a) What is the *difference* between the pressure of the blood in your brain when you stand on your head and the pressure when you stand on your feet? Assume that you are 1.85 m tall. The density of blood is 1060 kg/m^3. (b) What effect does the increased pressure have on the blood vessels in your brain?

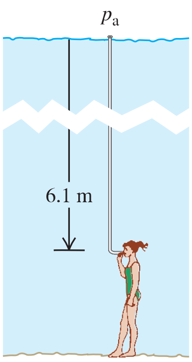
12.14 •• You are designing a diving bell to withstand the pressure of seawater at a depth of 250 m. (a) What is the gauge pressure at this depth? (You can ignore changes in the density of the water with depth.) (b) At this depth, what is the net force due to the water outside and the air inside the bell on a circular glass window 30.0 cm in diameter if the pressure inside the diving bell equals the pressure at the surface of the water? (You can ignore the small variation of pressure over the surface of the window.)

12.15 •• **BIO** **Ear Damage from Diving.** If the force on the tympanic membrane (eardrum) increases by about 1.5 N above the force from atmospheric pressure, the membrane can be damaged. When you go scuba diving in the ocean, below what depth could damage to your eardrum start to occur? The eardrum is typically 8.2 mm in diameter. (Consult Table 12.1.)

12.16 •• The liquid in the open-tube manometer in Fig. 12.8a is mercury, $y_1 = 3.00$ cm, and $y_2 = 7.00$ cm. Atmospheric pressure is 980 millibars. (a) What is the absolute pressure at the bottom of the U-shaped tube? (b) What is the absolute pressure in the open tube at a depth of 4.00 cm below the free surface? (c) What is the absolute pressure of the gas in the container? (d) What is the gauge pressure of the gas in pascals?

12.17 • **BIO** There is a maximum depth at which a diver can breathe through a snorkel tube (Fig. E12.17) because as the depth increases, so does the pressure difference, which tends to collapse the diver's lungs. Since the snorkel connects the air in the lungs to the atmosphere at the surface, the pressure inside the lungs is atmospheric pressure. What is the external–internal pressure difference when the diver's lungs are at a depth of 6.1 m (about 20 ft)? Assume that the diver is in freshwater. (A scuba diver breathing from compressed air tanks can operate at greater depths than can a snorkeler, since the pressure of the air inside the scuba diver's lungs increases to match the external pressure of the water.)

Figure **E12.17**



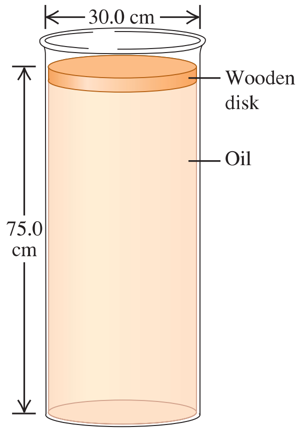
12.18 •• A tall cylinder with a cross-sectional area 12.0 cm^2 is partially filled with mercury; the surface of the mercury is 5.00 cm above the bottom of the cylinder. Water is slowly poured in on top of the mercury, and the two fluids don't mix. What volume of water must be added to double the gauge pressure at the bottom of the cylinder?

12.19 •• An electrical short cuts off all power to a submersible diving vehicle when it is 30 m below the surface of the ocean. The crew must push out a hatch of area 0.75 m^2 and weight 300 N on the bottom to escape. If the pressure inside is 1.0 atm, what downward force must the crew exert on the hatch to open it?

12.20 •• A closed container is partially filled with water. Initially, the air above the water is at atmospheric pressure (1.01×10^5 Pa) and the gauge pressure at the bottom of the water is 2500 Pa. Then additional air is pumped in, increasing the pressure of the air above the water by 1500 Pa. (a) What is the gauge pressure at the bottom of the water? (b) By how much must the water level in the container be reduced, by drawing some water out through a valve at the bottom of the container, to return the gauge pressure at the bottom of the water to its original value of 2500 Pa? The pressure of the air above the water is maintained at 1500 Pa above atmospheric pressure.

12.21 •• A cylindrical disk of wood weighing 45.0 N and having a diameter of 30.0 cm floats on a cylinder of oil of density 0.850 g/cm^3 (Fig. E12.21). The cylinder of oil is 75.0 cm deep and has a diameter the same as that of the wood. (a) What is the gauge pressure at the top of the oil column? (b) Suppose now that someone puts a weight of 83.0 N on top of the wood, but no oil seeps around the edge of the wood. What is the *change* in pressure at (i) the bottom of the oil and (ii) halfway down in the oil?

Figure **E12.21**



12.22 •• **Exploring Venus.** The surface pressure on Venus is 92 atm, and the acceleration due to gravity there is 0.894g. In a future exploratory mission, an upright cylindrical tank of benzene is sealed at the top but still pressurized at 92 atm just above the benzene. The tank has a diameter of 1.72 m, and the benzene column is 11.50 m tall. Ignore any effects due to the very high temperature on Venus. (a) What total force is exerted on the inside surface of the bottom of the tank? (b) What force does the Venusian atmosphere exert on the outside surface of the bottom of the tank? (c) What total inward force does the atmosphere exert on the vertical walls of the tank?

12.23 •• **Hydraulic Lift I.** For the hydraulic lift shown in Fig. 12.7, what must be the ratio of the diameter of the vessel at the car to the diameter of the vessel where the force F_1 is applied so that a 1520-kg car can be lifted with a force F_1 of just 125 N?

12.24 • **Hydraulic Lift II.** The piston of a hydraulic automobile lift is 0.30 m in diameter. What gauge pressure, in pascals, is required to lift a car with a mass of 1200 kg? Also express this pressure in atmospheres.

Section 12.3 Buoyancy

12.25 • A 950-kg cylindrical can buoy floats vertically in salt water. The diameter of the buoy is 0.900 m. Calculate the additional distance the buoy will sink when a 70.0-kg man stands on top of it.

12.26 •• A slab of ice floats on a freshwater lake. What minimum volume must the slab have for a 45.0-kg woman to be able to stand on it without getting her feet wet?

12.27 •• An ore sample weighs 17.50 N in air. When the sample is suspended by a light cord and totally immersed in water, the tension in the cord is 11.20 N. Find the total volume and the density of the sample.

12.28 •• You are preparing some apparatus for a visit to a newly discovered planet Caasi having oceans of glycerine and a surface acceleration due to gravity of 4.15 m/s^2. If your apparatus floats in the oceans on earth with 25.0% of its volume

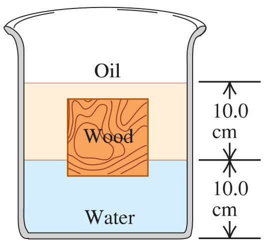
submerged, what percentage will be submerged in the glycerine oceans of Caasi?

12.29 •• An object of average density ρ floats at the surface of a fluid of density ρ_{fluid}. (a) How must the two densities be related? (b) In view of the answer to part (a), how can steel ships float in water? (c) In terms of ρ and ρ_{fluid}, what fraction of the object is submerged and what fraction is above the fluid? Check that your answers give the correct limiting behavior as $\rho \to \rho_{\text{fluid}}$ and as $\rho \to 0$. (d) While on board your yacht, your cousin Throckmorton cuts a rectangular piece (dimensions $5.0 \times 4.0 \times 3.0$ cm) out of a life preserver and throws it into the ocean. The piece has a mass of 42 g. As it floats in the ocean, what percentage of its volume is above the surface?

12.30 • A hollow plastic sphere is held below the surface of a freshwater lake by a cord anchored to the bottom of the lake. The sphere has a volume of 0.650 m^3 and the tension in the cord is 900 N. (a) Calculate the buoyant force exerted by the water on the sphere. (b) What is the mass of the sphere? (c) The cord breaks and the sphere rises to the surface. When the sphere comes to rest, what fraction of its volume will be submerged?

12.31 •• A cubical block of wood, 10.0 cm on a side, floats at the interface between oil and water with its lower surface 1.50 cm below the interface (Fig. E12.31). The density of the oil is 790 kg/m^3. (a) What is the gauge pressure at the upper face of the block? (b) What is the gauge pressure at the lower face of the block? (c) What are the mass and density of the block?

Figure **E12.31**



12.32 • A solid aluminum ingot weighs 89 N in air. (a) What is its volume? (b) The ingot is suspended from a rope and totally immersed in water. What is the tension in the rope (the *apparent* weight of the ingot in water)?

12.33 •• A rock is suspended by a light string. When the rock is in air, the tension in the string is 39.2 N. When the rock is totally immersed in water, the tension is 28.4 N. When the rock is totally immersed in an unknown liquid, the tension is 18.6 N. What is the density of the unknown liquid?

Section 12.4 Fluid Flow

12.34 •• Water runs into a fountain, filling all the pipes, at a steady rate of 0.750 m^3/s. (a) How fast will it shoot out of a hole 4.50 cm in diameter? (b) At what speed will it shoot out if the diameter of the hole is three times as large?

12.35 •• A shower head has 20 circular openings, each with radius 1.0 mm. The shower head is connected to a pipe with radius 0.80 cm. If the speed of water in the pipe is 3.0 m/s, what is its speed as it exits the shower-head openings?

12.36 • Water is flowing in a pipe with a varying cross-sectional area, and at all points the water completely fills the pipe. At point 1 the cross-sectional area of the pipe is 0.070 m^2, and the magnitude of the fluid velocity is 3.50 m/s. (a) What is the fluid speed at points in the pipe where the cross-sectional area is (a) 0.105 m^2 and (b) 0.047 m^2? (c) Calculate the volume of water discharged from the open end of the pipe in 1.00 hour.

12.37 • Water is flowing in a pipe with a circular cross section but with varying cross-sectional area, and at all points the water completely fills the pipe. (a) At one point in the pipe the radius is 0.150 m. What is the speed of the water at this point if water is flowing into this pipe at a steady rate of 1.20 m^3/s? (b) At a second point in the pipe the water speed is 3.80 m/s. What is the radius of the pipe at this point?

12.38 • **Home Repair.** You need to extend a 2.50-inch-diameter pipe, but you have only a 1.00-inch-diameter pipe on hand. You make a fitting to connect these pipes end to end. If the water is flowing at 6.00 cm/s in the wide pipe, how fast will it be flowing through the narrow one?

12.39 • At a point where an irrigation canal having a rectangular cross section is 18.5 m wide and 3.75 m deep, the water flows at 2.50 cm/s. At a point downstream, but on the same level, the canal is 16.5 m wide, but the water flows at 11.0 cm/s. How deep is the canal at this point?

12.40 •• BIO **Artery Blockage.** A medical technician is trying to determine what percentage of a patient's artery is blocked by plaque. To do this, she measures the blood pressure just before the region of blockage and finds that it is 1.20×10^4 Pa, while in the region of blockage it is 1.15×10^4 Pa. Furthermore, she knows that blood flowing through the normal artery just before the point of blockage is traveling at 30.0 cm/s, and the specific gravity of this patient's blood is 1.06. What percentage of the cross-sectional area of the patient's artery is blocked by the plaque?

Section 12.5 Bernoulli's Equation

12.41 •• A sealed tank containing seawater to a height of 11.0 m also contains air above the water at a gauge pressure of 3.00 atm. Water flows out from the bottom through a small hole. How fast is this water moving?

12.42 • A small circular hole 6.00 mm in diameter is cut in the side of a large water tank, 14.0 m below the water level in the tank. The top of the tank is open to the air. Find (a) the speed of efflux of the water and (b) the volume discharged per second.

12.43 • What gauge pressure is required in the city water mains for a stream from a fire hose connected to the mains to reach a vertical height of 15.0 m? (Assume that the mains have a much larger diameter than the fire hose.)

12.44 •• At one point in a pipeline the water's speed is 3.00 m/s and the gauge pressure is 5.00×10^4 Pa. Find the gauge pressure at a second point in the line, 11.0 m lower than the first, if the pipe diameter at the second point is twice that at the first.

12.45 • At a certain point in a horizontal pipeline, the water's speed is 2.50 m/s and the gauge pressure is 1.80×10^4 Pa. Find the gauge pressure at a second point in the line if the cross-sectional area at the second point is twice that at the first.

12.46 • A soft drink (mostly water) flows in a pipe at a beverage plant with a mass flow rate that would fill 220 0.355-L cans per minute. At point 2 in the pipe, the gauge pressure is 152 kPa and the cross-sectional area is 8.00 cm^2. At point 1, 1.35 m above point 2, the cross-sectional area is 2.00 cm^2. Find the (a) mass flow rate; (b) volume flow rate; (c) flow speeds at points 1 and 2; (d) gauge pressure at point 1.

12.47 •• A golf course sprinkler system discharges water from a horizontal pipe at the rate of 7200 cm^3/s. At one point in the pipe, where the radius is 4.00 cm, the water's absolute pressure is 2.40×10^5 Pa. At a second point in the pipe, the water passes through a constriction where the radius is 2.00 cm. What is the water's absolute pressure as it flows through this constriction?

Section 12.6 Viscosity and Turbulence

12.48 • A pressure difference of 6.00×10^4 Pa is required to maintain a volume flow rate of 0.800 m^3/s for a viscous fluid flowing through a section of cylindrical pipe that has radius 0.210 m.

What pressure difference is required to maintain the same volume flow rate if the radius of the pipe is decreased to 0.0700 m?

12.49 •• BIO **Clogged Artery.** Viscous blood is flowing through an artery partially clogged by cholesterol. A surgeon wants to remove enough of the cholesterol to double the flow rate of blood through this artery. If the original diameter of the artery is D, what should be the new diameter (in terms of D) to accomplish this for the same pressure gradient?

PROBLEMS

12.50 •• CP The deepest point known in any of the earth's oceans is in the Marianas Trench, 10.92 km deep. (a) Assuming water is incompressible, what is the pressure at this depth? Use the density of seawater. (b) The actual pressure is 1.16×10^8 Pa; your calculated value will be less because the density actually varies with depth. Using the compressibility of water and the actual pressure, find the density of the water at the bottom of the Marianas Trench. What is the percent change in the density of the water?

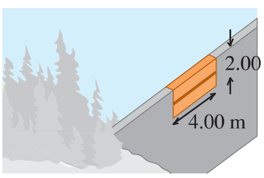
12.51 ••• In a lecture demonstration, a professor pulls apart two hemispherical steel shells (diameter D) with ease using their attached handles. She then places them together, pumps out the air to an absolute pressure of p, and hands them to a bodybuilder in the back row to pull apart. (a) If atmospheric pressure is p_0, how much force must the bodybuilder exert on each shell? (b) Evaluate your answer for the case $p = 0.025$ atm, $D = 10.0$ cm.

12.52 •• BIO **Fish Navigation.** (a) As you can tell by watching them in an aquarium, fish are able to remain at any depth in water with no effort. What does this ability tell you about their density? (b) Fish are able to inflate themselves using a sac (called the *swim bladder*) located under their spinal column. These sacs can be filled with an oxygen–nitrogen mixture that comes from the blood. If a 2.75-kg fish in freshwater inflates itself and increases its volume by 10%, find the *net* force that the *water* exerts on it. (c) What is the net *external* force on it? Does the fish go up or down when it inflates itself?

12.53 ••• CALC A swimming pool is 5.0 m long, 4.0 m wide, and 3.0 m deep. Compute the force exerted by the water against (a) the bottom and (b) either end. (*Hint:* Calculate the force on a thin, horizontal strip at a depth h, and integrate this over the end of the pool.) Do not include the force due to air pressure.

12.54 ••• CP CALC The upper edge of a gate in a dam runs along the water surface. The gate is 2.00 m high and 4.00 m wide and is hinged along a horizontal line through its center (Fig. P12.54). Calculate the torque about the hinge arising from the force due to the water. (*Hint:* Use a procedure similar to that used in Problem 12.53; calculate the torque on a thin, horizontal strip at a depth h and integrate this over the gate.)

Figure **P12.54**



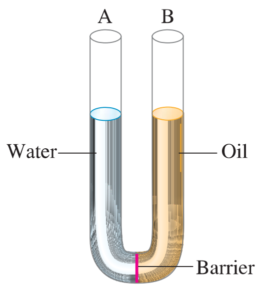
12.55 ••• CP CALC **Force and Torque on a Dam.** A dam has the shape of a rectangular solid. The side facing the lake has area A and height H. The surface of the freshwater lake behind the dam is at the top of the dam. (a) Show that the net horizontal force exerted by the water on the dam equals $\frac{1}{2}\rho g H A$—that is, the average gauge pressure across the face of the dam times the area (see Problem 12.53). (b) Show that the torque exerted by the water about an axis along the bottom of the dam is $\rho g H^2 A/6$. (c) How do the force and torque depend on the size of the lake?

12.56 •• **Ballooning on Mars.** It has been proposed that we could explore Mars using inflated balloons to hover just above the surface. The buoyancy of the atmosphere would keep the balloon aloft. The density of the Martian atmosphere is $0.0154\ \text{kg/m}^3$ (although this varies with temperature). Suppose we construct these balloons of a thin but tough plastic having a density such that each square meter has a mass of 5.00 g. We inflate them with a very light gas whose mass we can neglect. (a) What should be the radius and mass of these balloons so they just hover above the surface of Mars? (b) If we released one of the balloons from part (a) on earth, where the atmospheric density is $1.20\ \text{kg/m}^3$, what would be its initial acceleration assuming it was the same size as on Mars? Would it go up or down? (c) If on Mars these balloons have five times the radius found in part (a), how heavy an instrument package could they carry?

12.57 •• A 0.180-kg cube of ice (frozen water) is floating in glycerine. The gylcerine is in a tall cylinder that has inside radius 3.50 cm. The level of the glycerine is well below the top of the cylinder. If the ice completely melts, by what distance does the height of liquid in the cylinder change? Does the level of liquid rise or fall? That is, is the surface of the water above or below the original level of the gylcerine before the ice melted?

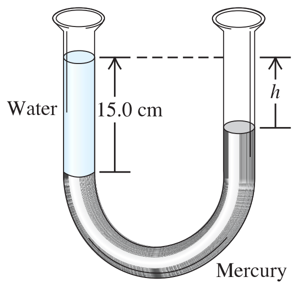
12.58 •• A narrow, U-shaped glass tube with open ends is filled with 25.0 cm of oil (of specific gravity 0.80) and 25.0 cm of water on opposite sides, with a barrier separating the liquids (Fig. P12.58). (a) Assume that the two liquids do not mix, and find the final heights of the columns of liquid in each side of the tube after the barrier is removed. (b) For the following cases, arrive at your answer by simple physical reasoning, not by calculations: (i) What would be the height on each side if the oil and water had equal densities? (ii) What would the heights be if the oil's density were much less than that of water?

Figure **P12.58**



12.59 • A U-shaped tube open to the air at both ends contains some mercury. A quantity of water is carefully poured into the left arm of the U-shaped tube until the vertical height of the water column is 15.0 cm (Fig. P12.59). (a) What is the gauge pressure at the water–mercury interface? (b) Calculate the vertical distance h from the top of the mercury in the right-hand arm of the tube to the top of the water in the left-hand arm.

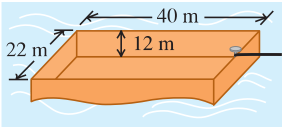
Figure **P12.59**



12.60 •• CALC **The Great Molasses Flood.** On the afternoon of January 15, 1919, an unusually warm day in Boston, a 17.7-m-high, 27.4-m-diameter cylindrical metal tank used for storing molasses ruptured. Molasses flooded into the streets in a 5-m-deep stream, killing pedestrians and horses and knocking down buildings. The molasses had a density of $1600\ \text{kg/m}^3$. If the tank was full before the accident, what was the total outward force the molasses exerted on its sides? (*Hint:* Consider the outward force on a circular ring of the tank wall of width dy and at a depth y below the surface. Integrate to find the total outward force. Assume that before the tank ruptured, the pressure at the surface of the molasses was equal to the air pressure outside the tank.)

12.61 • An open barge has the dimensions shown in Fig. P12.61. If the barge is made out of 4.0-cm-thick steel plate on each of its four sides and its bottom, what mass of coal can the barge carry in freshwater without sinking? Is there enough room in the barge to hold this amount of coal? (The density of coal is about 1500 kg/m^3.)

Figure **P12.61**



12.62 ••• A hot-air balloon has a volume of 2200 m^3. The balloon fabric (the envelope) weighs 900 N. The basket with gear and full propane tanks weighs 1700 N. If the balloon can barely lift an additional 3200 N of passengers, breakfast, and champagne when the outside air density is 1.23 kg/m^3, what is the average density of the heated gases in the envelope?

12.63 •• Advertisements for a certain small car claim that it floats in water. (a) If the car's mass is 900 kg and its interior volume is 3.0 m^3, what fraction of the car is immersed when it floats? You can ignore the volume of steel and other materials. (b) Water gradually leaks in and displaces the air in the car. What fraction of the interior volume is filled with water when the car sinks?

12.64 • A single ice cube with mass 9.70 g floats in a glass completely full of 420 cm^3 of water. You can ignore the water's surface tension and its variation in density with temperature (as long as it remains a liquid). (a) What volume of water does the ice cube displace? (b) When the ice cube has completely melted, has any water overflowed? If so, how much? If not, explain why this is so. (c) Suppose the water in the glass had been very salty water of density 1050 kg/m^3. What volume of salt water would the 9.70-g ice cube displace? (d) Redo part (b) for the freshwater ice cube in the salty water.

12.65 ••• A piece of wood is 0.600 m long, 0.250 m wide, and 0.080 m thick. Its density is 700 kg/m^3. What volume of lead must be fastened underneath it to sink the wood in calm water so that its top is just even with the water level? What is the mass of this volume of lead?

12.66 •• A hydrometer consists of a spherical bulb and a cylindrical stem with a cross-sectional area of 0.400 cm^2 (see Fig. 12.12a). The total volume of bulb and stem is 13.2 cm^3. When immersed in water, the hydrometer floats with 8.00 cm of the stem above the water surface. When the hydrometer is immersed in an organic fluid, 3.20 cm of the stem is above the surface. Find the density of the organic fluid. (*Note:* This illustrates the precision of such a hydrometer. Relatively small density differences give rise to relatively large differences in hydrometer readings.)

12.67 •• The densities of air, helium, and hydrogen (at $p = 1.0$ atm and $T = 20°\text{C}$) are 1.20 kg/m^3, 0.166 kg/m^3, and 0.0899 kg/m^3, respectively. (a) What is the volume in cubic meters displaced by a hydrogen-filled airship that has a total "lift" of 90.0 kN? (The "lift" is the amount by which the buoyant force exceeds the weight of the gas that fills the airship.) (b) What would be the "lift" if helium were used instead of hydrogen? In view of your answer, why is helium used in modern airships like advertising blimps?

12.68 •• When an open-faced boat has a mass of 5750 kg, including its cargo and passengers, it floats with the water just up to the top of its gunwales (sides) on a freshwater lake. (a) What is the volume of this boat? (b) The captain decides that it is too dangerous to float with his boat on the verge of sinking, so he decides to throw some cargo overboard so that 20% of the boat's volume will be above water. How much mass should he throw out?

12.69 •• **CP** An open cylindrical tank of acid rests at the edge of a table 1.4 m above the floor of the chemistry lab. If this tank springs a small hole in the side at its base, how far from the foot of the table will the acid hit the floor if the acid in the tank is 75 cm deep?

12.70 •• **CP** A firehose must be able to shoot water to the top of a building 28.0 m tall when aimed straight up. Water enters this hose at a steady rate of 0.500 m^3/s and shoots out of a round nozzle. (a) What is the maximum diameter this nozzle can have? (b) If the only nozzle available has a diameter twice as great, what is the highest point the water can reach?

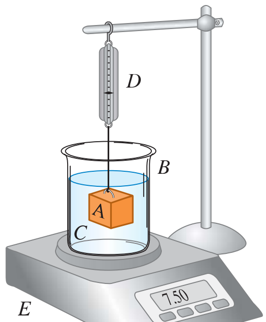
12.71 •• **CP** You drill a small hole in the side of a vertical cylindrical water tank that is standing on the ground with its top open to the air. (a) If the water level has a height H, at what height above the base should you drill the hole for the water to reach its greatest distance from the base of the cylinder when it hits the ground? (b) What is the greatest distance the water will reach?

12.72 ••• **CALC** A closed and elevated vertical cylindrical tank with diameter 2.00 m contains water to a depth of 0.800 m. A worker accidently pokes a circular hole with diameter 0.0200 m in the bottom of the tank. As the water drains from the tank, compressed air above the water in the tank maintains a gauge pressure of 5.00×10^3 Pa at the surface of the water. Ignore any effects of viscosity. (a) Just after the hole is made, what is the speed of the water as it emerges from the hole? What is the ratio of this speed to the efflux speed if the top of the tank is open to the air? (b) How much time does it take for all the water to drain from the tank? What is the ratio of this time to the time it takes for the tank to drain if the top of the tank is open to the air?

12.73 •• A block of balsa wood placed in one scale pan of an equal-arm balance is exactly balanced by a 0.115-kg brass mass in the other scale pan. Find the true mass of the balsa wood if its density is 150 kg/m^3. Explain why it is accurate to ignore the buoyancy in air of the brass but *not* the buoyancy in air of the balsa wood.

12.74 •• Block A in Fig. P12.74 hangs by a cord from spring balance D and is submerged in a liquid C contained in beaker B. The mass of the beaker is 1.00 kg; the mass of the liquid is 1.80 kg. Balance D reads 3.50 kg, and balance E reads 7.50 kg. The volume of block A is 3.80×10^{-3} m^3. (a) What is the density of the liquid? (b) What will each balance read if block A is pulled up out of the liquid?

Figure **P12.74**



12.75 •• A hunk of aluminum is completely covered with a gold shell to form an ingot of weight 45.0 N. When you suspend the ingot from a spring balance and submerge the ingot in water, the balance reads 39.0 N. What is the weight of the gold in the shell?

12.76 •• A plastic ball has radius 12.0 cm and floats in water with 24.0% of its volume submerged. (a) What force must you apply to the ball to hold it at rest totally below the surface of the water? (b) If you let go of the ball, what is its acceleration the instant you release it?

12.77 •• The weight of a king's solid crown is w. When the crown is suspended by a light rope and completely immersed in water, the tension in the rope (the crown's apparent weight) is fw. (a) Prove that the crown's relative density (specific gravity) is $1/(1 - f)$. Discuss the meaning of the limits as f approaches 0 and 1. (b) If the crown is solid gold and weighs 12.9 N in air, what is its apparent

weight when completely immersed in water? (c) Repeat part (b) if the crown is solid lead with a very thin gold plating, but still has a weight in air of 12.9 N.

12.78 •• A piece of steel has a weight w, an apparent weight (see Problem 12.77) w_{water} when completely immersed in water, and an apparent weight w_{fluid} when completely immersed in an unknown fluid. (a) Prove that the fluid's density relative to water (specific gravity) is $(w - w_{fluid})/(w - w_{water})$. (b) Is this result reasonable for the three cases of w_{fluid} greater than, equal to, or less than w_{water}? (c) The apparent weight of the piece of steel in water of density $1000\ kg/m^3$ is 87.2% of its weight. What percentage of its weight will its apparent weight be in formic acid (density $1220\ kg/m^3$)?

12.79 ••• You cast some metal of density ρ_m in a mold, but you are worried that there might be cavities within the casting. You measure the weight of the casting to be w, and the buoyant force when it is completely surrounded by water to be B. (a) Show that $V_0 = B/(\rho_{water}g) - w/(\rho_m g)$ is the total volume of any enclosed cavities. (b) If your metal is copper, the casting's weight is 156 N, and the buoyant force is 20 N, what is the total volume of any enclosed cavities in your casting? What fraction is this of the total volume of the casting?

12.80 • A cubical block of wood 0.100 m on a side and with a density of $550\ kg/m^3$ floats in a jar of water. Oil with a density of $750\ kg/m^3$ is poured on the water until the top of the oil layer is 0.035 m below the top of the block. (a) How deep is the oil layer? (b) What is the gauge pressure at the block's lower face?

12.81 •• **Dropping Anchor.** An iron anchor with mass 35.0 kg and density $7860\ kg/m^3$ lies on the deck of a small barge that has vertical sides and floats in a freshwater river. The area of the bottom of the barge is $8.00\ m^2$. The anchor is thrown overboard but is suspended above the bottom of the river by a rope; the mass and volume of the rope are small enough to ignore. After the anchor is overboard and the barge has finally stopped bobbing up and down, has the barge risen or sunk down in the water? By what vertical distance?

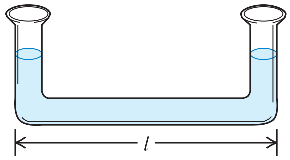
12.82 •• Assume that crude oil from a supertanker has density $750\ kg/m^3$. The tanker runs aground on a sandbar. To refloat the tanker, its oil cargo is pumped out into steel barrels, each of which has a mass of 15.0 kg when empty and holds $0.120\ m^3$ of oil. You can ignore the volume occupied by the steel from which the barrel is made. (a) If a salvage worker accidentally drops a filled, sealed barrel overboard, will it float or sink in the seawater? (b) If the barrel floats, what fraction of its volume will be above the water surface? If it sinks, what minimum tension would have to be exerted by a rope to haul the barrel up from the ocean floor? (c) Repeat parts (a) and (b) if the density of the oil is $910\ kg/m^3$ and the mass of each empty barrel is 32.0 kg.

12.83 ••• A cubical block of density ρ_B and with sides of length L floats in a liquid of greater density ρ_L. (a) What fraction of the block's volume is above the surface of the liquid? (b) The liquid is denser than water (density ρ_W) and does not mix with it. If water is poured on the surface of the liquid, how deep must the water layer be so that the water surface just rises to the top of the block? Express your answer in terms of L, ρ_B, ρ_L, and ρ_W. (c) Find the depth of the water layer in part (b) if the liquid is mercury, the block is made of iron, and the side length is 10.0 cm.

12.84 •• A barge is in a rectangular lock on a freshwater river. The lock is 60.0 m long and 20.0 m wide, and the steel doors on each end are closed. With the barge floating in the lock, a 2.50×10^6 N load of scrap metal is put onto the barge. The metal has density $9000\ kg/m^3$. (a) When the load of scrap metal, initially on the bank, is placed onto the barge, what vertical distance does the water in the lock rise? (b) The scrap metal is now pushed overboard into the water. Does the water level in the lock rise, fall, or remain the same? If it rises or falls, by what vertical distance does it change?

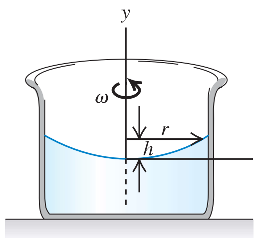
12.85 • **CP CALC** A U-shaped tube with a horizontal portion of length l (Fig. P12.85) contains a liquid. What is the difference in height between the liquid columns in the vertical arms (a) if the tube has an acceleration a toward the right and (b) if the tube is mounted on a horizontal turntable rotating with an angular speed ω with one of the vertical arms on the axis of rotation? (c) Explain why the difference in height does not depend on the density of the liquid or on the cross-sectional area of the tube. Would it be the same if the vertical tubes did not have equal cross-sectional areas? Would it be the same if the horizontal portion were tapered from one end to the other? Explain.

Figure **P12.85**



12.86 • **CP CALC** A cylindrical container of an incompressible liquid with density ρ rotates with constant angular speed ω about its axis of symmetry, which we take to be the y-axis (Fig. P12.86). (a) Show that the pressure at a given height within the fluid increases in the radial direction (outward from the axis of rotation) according to $\partial p/\partial r = \rho\omega^2 r$. (b) Integrate this partial differential equation to find the pressure as a function of distance from the axis of rotation along a horizontal line at $y = 0$. (c) Combine the result of part (b) with Eq. (12.5) to show that the surface of the rotating liquid has a *parabolic* shape; that is, the height of the liquid is given by $h(r) = \omega^2 r^2/2g$. (This technique is used for making parabolic telescope mirrors; liquid glass is rotated and allowed to solidify while rotating.)

Figure **P12.86**



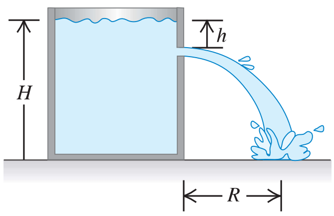
12.87 •• **CP CALC** An incompressible fluid with density ρ is in a horizontal test tube of inner cross-sectional area A. The test tube spins in a horizontal circle in an ultracentrifuge at an angular speed ω. Gravitational forces are negligible. Consider a volume element of the fluid of area A and thickness dr' a distance r' from the rotation axis. The pressure on its inner surface is p and on its outer surface is $p + dp$. (a) Apply Newton's second law to the volume element to show that $dp = \rho\omega^2 r'\,dr'$. (b) If the surface of the fluid is at a radius r_0 where the pressure is p_0, show that the pressure p at a distance $r \geq r_0$ is $p = p_0 + \rho\omega^2(r^2 - r_0^2)/2$. (c) An object of volume V and density ρ_{ob} has its center of mass at a distance R_{cmob} from the axis. Show that the net horizontal force on the object is $\rho V\omega^2 R_{cm}$, where R_{cm} is the distance from the axis to the center of mass of the displaced fluid. (d) Explain why the object will move inward if $\rho R_{cm} > \rho_{ob} R_{cmob}$ and outward if $\rho R_{cm} < \rho_{ob} R_{cmob}$. (e) For small objects of uniform density, $R_{cm} = R_{cmob}$. What happens to a mixture of small objects of this kind with different densities in an ultracentrifuge?

12.88 ••• **CALC** Untethered helium balloons, floating in a car that has all the windows rolled up and outside air vents closed, move in the direction of the car's acceleration, but loose balloons filled with air move in the opposite direction. To show why, consider only the horizontal forces acting on the balloons. Let a be the magnitude of the car's forward acceleration. Consider a horizontal tube of air with a cross-sectional area A that extends from the

windshield, where $x = 0$ and $p = p_0$, back along the x-axis. Now consider a volume element of thickness dx in this tube. The pressure on its front surface is p and the pressure on its rear surface is $p + dp$. Assume the air has a constant density ρ. (a) Apply Newton's second law to the volume element to show that $dp = \rho a\, dx$. (b) Integrate the result of part (a) to find the pressure at the front surface in terms of a and x. (c) To show that considering ρ constant is reasonable, calculate the pressure difference in atm for a distance as long as 2.5 m and a large acceleration of 5.0 m/s^2. (d) Show that the net horizontal force on a balloon of volume V is ρVa. (e) For negligible friction forces, show that the acceleration of the balloon (average density ρ_{bal}) is $(\rho/\rho_{bal})a$, so that the acceleration relative to the car is $a_{rel} = [(\rho/\rho_{bal}) - 1]a$. (f) Use the expression for a_{rel} in part (e) to explain the movement of the balloons.

12.89 • **CP** Water stands at a depth H in a large, open tank whose side walls are vertical (Fig. P12.89). A hole is made in one of the walls at a depth h below the water surface. (a) At what distance R from the foot of the wall does the emerging stream strike the floor? (b) How far above the bottom of the tank could a second hole be cut so that the stream emerging from it could have the same range as for the first hole?

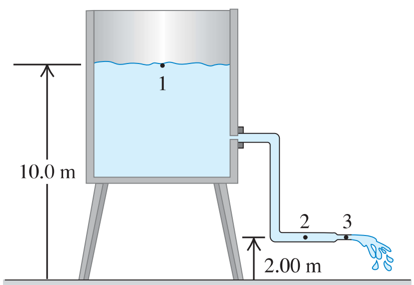
Figure **P12.89**



12.90 ••• A cylindrical bucket, open at the top, is 25.0 cm high and 10.0 cm in diameter. A circular hole with a cross-sectional area 1.50 cm^2 is cut in the center of the bottom of the bucket. Water flows into the bucket from a tube above it at the rate of 2.40×10^{-4} m^3/s. How high will the water in the bucket rise?

12.91 • Water flows steadily from an open tank as in Fig. P12.91. The elevation of point 1 is 10.0 m, and the elevation of points 2 and 3 is 2.00 m. The cross-sectional area at point 2 is 0.0480 m^2; at point 3 it is 0.0160 m^2. The area of the tank is very large compared with the cross-sectional area of the pipe. Assuming that Bernoulli's equation applies, compute (a) the discharge rate in cubic meters per second and (b) the gauge pressure at point 2.

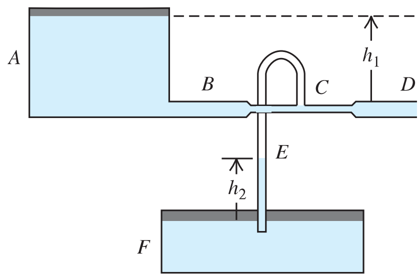
Figure **P12.91**



12.92 •• **CP** In 1993 the radius of Hurricane Emily was about 350 km. The wind speed near the center ("eye") of the hurricane, whose radius was about 30 km, reached about 200 km/h. As air swirled in from the rim of the hurricane toward the eye, its angular momentum remained roughly constant. (a) Estimate the wind speed at the rim of the hurricane. (b) Estimate the pressure difference at the earth's surface between the eye and the rim. (*Hint:* See Table 12.1.) Where is the pressure greater? (c) If the kinetic energy of the swirling air in the eye could be converted completely to gravitational potential energy, how high would the air go? (d) In fact, the air in the eye is lifted to heights of several kilometers. How can you reconcile this with your answer to part (c)?

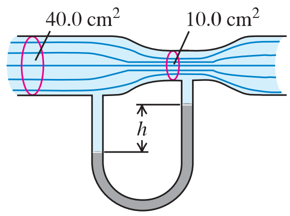
12.93 •• Two very large open tanks A and F (Fig. P12.93) contain the same liquid. A horizontal pipe BCD, having a constriction at C and open to the air at D, leads out of the bottom of tank A, and a vertical pipe E opens into the constriction at C and dips into the liquid in tank F. Assume streamline flow and no viscosity. If the cross-sectional area at C is one-half the area at D and if D is a distance h_1 below the level of the liquid in A, to what height h_2 will liquid rise in pipe E? Express your answer in terms of h_1.

Figure **P12.93**



12.94 •• The horizontal pipe shown in Fig. P12.94 has a cross-sectional area of 40.0 cm^2 at the wider portions and 10.0 cm^2 at the constriction. Water is flowing in the pipe, and the discharge from the pipe is 6.00×10^{-3} m^3/s (6.00 L/s). Find (a) the flow speeds at the wide and the narrow portions; (b) the pressure difference between these portions; (c) the difference in height between the mercury columns in the U-shaped tube.

Figure **P12.94**



12.95 • A liquid flowing from a vertical pipe has a definite shape as it flows from the pipe. To get the equation for this shape, assume that the liquid is in free fall once it leaves the pipe. Just as it leaves the pipe, the liquid has speed v_0 and the radius of the stream of liquid is r_0. (a) Find an equation for the speed of the liquid as a function of the distance y it has fallen. Combining this with the equation of continuity, find an expression for the radius of the stream as a function of y. (b) If water flows out of a vertical pipe at a speed of 1.20 m/s, how far below the outlet will the radius be one-half the original radius of the stream?

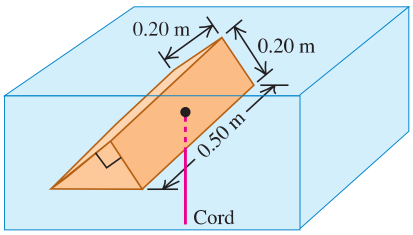
Challenge Problems

12.96 ••• A rock with mass $m = 3.00$ kg is suspended from the roof of an elevator by a light cord. The rock is totally immersed in a bucket of water that sits on the floor of the elevator, but the rock doesn't touch the bottom or sides of the bucket. (a) When the elevator is at rest, the tension in the cord is 21.0 N. Calculate the volume of the rock. (b) Derive an expression for the tension in the cord when the elevator is accelerating *upward* with an acceleration of magnitude a. Calculate the tension when $a = 2.50$ m/s^2

upward. (c) Derive an expression for the tension in the cord when the elevator is accelerating *downward* with an acceleration of magnitude a. Calculate the tension when $a = 2.50\ \mathrm{m/s^2}$ downward. (d) What is the tension when the elevator is in free fall with a downward acceleration equal to g?

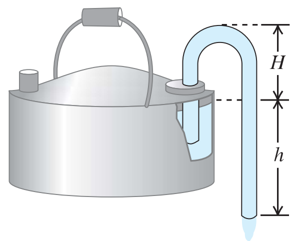
12.97 ••• **CALC** Suppose a piece of styrofoam, $\rho = 180\ \mathrm{kg/m^3}$, is held completely submerged in water (Fig. P12.97). (a) What is the tension in the cord? Find this using Archimedes's principle. (b) Use $p = p_0 + \rho g h$ to calculate directly the force exerted by the water on the two sloped sides and the bottom of the styrofoam; then show that the vector sum of these forces is the buoyant force.

Figure **P12.97**



12.98 ••• A *siphon,* as shown in Fig. P12.98, is a convenient device for removing liquids from containers. To establish the flow, the tube must be initially filled with fluid. Let the fluid have density ρ, and let the atmospheric pressure be p_{atm}. Assume that the cross-sectional area of the tube is the same at all points along it. (a) If the lower end of the siphon is at a distance h below the surface of the liquid in the container, what is the speed of the fluid as it flows out the lower end of the siphon? (Assume that the container has a very large diameter, and ignore any effects of viscosity.) (b) A curious feature of a siphon is that the fluid initially flows "uphill." What is the greatest height H that the high point of the tube can have if flow is still to occur?

Figure **P12.98**



Answers

Chapter Opening Question ?

The flesh of both the shark and the tropical fish is denser than seawater, so left to themselves they would sink. However, a tropical fish has a gas-filled body cavity called a swimbladder, so that the *average* density of the fish's body is the same as that of seawater and the fish neither sinks nor rises. Sharks have no such cavity. Hence they must swim constantly to keep from sinking, using their pectoral fins to provide lift much like the wings of an airplane (see Section 12.5).

Test Your Understanding Questions

12.1 Answer: (ii), (iv), (i) and (iii) (tie), (v) In each case the average density equals the mass divided by the volume. Hence we have (i) $\rho = (4.00\ \mathrm{kg})/(1.60 \times 10^{-3}\ \mathrm{m^3}) = 2.50 \times 10^3\ \mathrm{kg/m^3}$; (ii) $\rho = (8.00\ \mathrm{kg})/(1.60 \times 10^{-3}\ \mathrm{m^3}) = 5.00 \times 10^3\ \mathrm{kg/m^3}$; (iii) $\rho = (8.00\ \mathrm{kg})/(3.20 \times 10^{-3}\ \mathrm{m^3}) = 2.50 \times 10^3\ \mathrm{kg/m^3}$; (iv) $\rho = (2560\ \mathrm{kg})/(0.640\ \mathrm{m^3}) = 4.00 \times 10^3\ \mathrm{kg/m^3}$; (v) $\rho = (2560\ \mathrm{kg})/(1.28\ \mathrm{m^3}) = 2.00 \times 10^3\ \mathrm{kg/m^3}$. Note that compared to object (i), object (ii) has double the mass but the same volume and so has double the average density. Object (iii) has double the mass and double the volume of object (i), so (i) and (iii) have the same average density. Finally, object (v) has the same mass as object (iv) but double the volume, so (v) has half the average density of (iv).

12.2 Answer: (ii) From Eq. (12.9), the pressure outside the barometer is equal to the product $\rho g h$. When the barometer is taken out of the refrigerator, the density ρ decreases while the height h of the mercury column remains the same. Hence the air pressure must be lower outdoors than inside the refrigerator.

12.3 Answer: (i) Consider the water, the statue, and the container together as a system; the total weight of the system does not depend on whether the statue is immersed. The total supporting force, including the tension T and the upward force F of the scale on the container (equal to the scale reading), is the same in both cases. But we saw in Example 12.5 that T decreases by 7.84 N when the statue is immersed, so the scale reading F must *increase* by 7.84 N. An alternative viewpoint is that the water exerts an upward buoyant force of 7.84 N on the statue, so the statue must exert an equal downward force on the water, making the scale reading 7.84 N greater than the weight of water and container.

12.4 Answer: (ii) A highway that narrows from three lanes to one is like a pipe whose cross-sectional area narrows to one-third of its value. If cars behaved like the molecules of an incompressible fluid, then as the cars encountered the one-lane section, the spacing between cars (the "density") would stay the same but the cars would triple their speed. This would keep the "volume flow rate" (number of cars per second passing a point on the highway) the same. In real life cars behave like the molecules of a *compressible* fluid: They end up packed closer (the "density" increases) and fewer cars per second pass a point on the highway (the "volume flow rate" decreases).

12.5 Answer: (ii) Newton's second law tells us that a body accelerates (its velocity changes) in response to a net force. In fluid flow, a pressure difference between two points means that fluid particles moving between those two points experience a force, and this force causes the fluid particles to accelerate and change speed.

12.6 Answer: (iv) The required pressure is proportional to $1/R^4$, where R is the inside radius of the needle (half the inside diameter). With the smaller-diameter needle, the pressure is greater by a factor of $[(0.60\ \mathrm{mm})/(0.30\ \mathrm{mm})]^4 = 2^4 = 16$.

Bridging Problem

Answers: (a) $y = H - \left(\dfrac{d}{D}\right)^2 \sqrt{2gH}\, t + \left(\dfrac{d}{D}\right)^4 \dfrac{gt^2}{2}$

(b) $T = \sqrt{\dfrac{2H}{g}} \left(\dfrac{D}{d}\right)^2$ **(c)** $\sqrt{2}$

13 GRAVITATION

LEARNING GOALS

By studying this chapter, you will learn:

- How to calculate the gravitational forces that any two bodies exert on each other.
- How to relate the weight of an object to the general expression for gravitational force.
- How to use and interpret the generalized expression for gravitational potential energy.
- How to relate the speed, orbital period, and mechanical energy of a satellite in a circular orbit.
- The laws that describe the motions of planets, and how to work with these laws.
- What black holes are, how to calculate their properties, and how they are discovered.

? The rings of Saturn are made of countless individual orbiting particles. Do all the ring particles orbit at the same speed, or do the inner particles orbit faster or slower than the outer ones?

Some of the earliest investigations in physical science started with questions that people asked about the night sky. Why doesn't the moon fall to earth? Why do the planets move across the sky? Why doesn't the earth fly off into space rather than remaining in orbit around the sun? The study of gravitation provides the answers to these and many related questions.

As we remarked in Chapter 5, gravitation is one of the four classes of interactions found in nature, and it was the earliest of the four to be studied extensively. Newton discovered in the 17th century that the same interaction that makes an apple fall out of a tree also keeps the planets in their orbits around the sun. This was the beginning of *celestial mechanics,* the study of the dynamics of objects in space. Today, our knowledge of celestial mechanics allows us to determine how to put a satellite into any desired orbit around the earth or to choose just the right trajectory to send a spacecraft to another planet.

In this chapter you will learn the basic law that governs gravitational interactions. This law is *universal:* Gravity acts in the same fundamental way between the earth and your body, between the sun and a planet, and between a planet and one of its moons. We'll apply the law of gravitation to phenomena such as the variation of weight with altitude, the orbits of satellites around the earth, and the orbits of planets around the sun.

13.1 Newton's Law of Gravitation

The example of gravitational attraction that's probably most familiar to you is your *weight,* the force that attracts you toward the earth. During his study of the motions of the planets and of the moon, Newton discovered the fundamental character of the gravitational attraction between *any* two bodies. Along with his

three laws of motion, Newton published the **law of gravitation** in 1687. It may be stated as follows:

Every particle of matter in the universe attracts every other particle with a force that is directly proportional to the product of the masses of the particles and inversely proportional to the square of the distance between them.

Translating this into an equation, we have

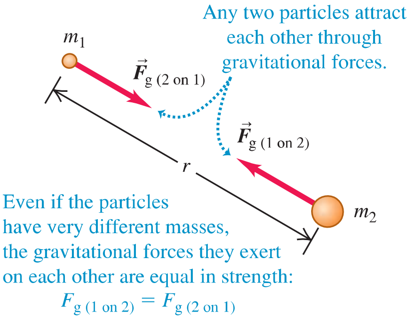
$$F_g = \frac{Gm_1m_2}{r^2} \quad \text{(law of gravitation)} \tag{13.1}$$

where F_g is the magnitude of the gravitational force on either particle, m_1 and m_2 are their masses, r is the distance between them (Fig. 13.1), and G is a fundamental physical constant called the **gravitational constant.** The numerical value of G depends on the system of units used.

Equation (13.1) tells us that the gravitational force between two particles decreases with increasing distance r: If the distance is doubled, the force is only one-fourth as great, and so on. Although many of the stars in the night sky are far more massive than the sun, they are so far away that their gravitational force on the earth is negligibly small.

CAUTION **Don't confuse g and G** Because the symbols g and G are so similar, it's common to confuse the two very different gravitational quantities that these symbols represent. Lowercase g is the acceleration due to gravity, which relates the weight w of a body to its mass m: $w = mg$. The value of g is different at different locations on the earth's surface and on the surfaces of different planets. By contrast, capital G relates the gravitational force between any two bodies to their masses and the distance between them. We call G a *universal* constant because it has the same value for any two bodies, no matter where in space they are located. In the next section we'll see how the values of g and G are related.

13.1 The gravitational forces between two particles of masses m_1 and m_2.



Gravitational forces always act along the line joining the two particles, and they form an action–reaction pair. Even when the masses of the particles are different, the two interaction forces have equal magnitude (Fig. 13.1). The attractive force that your body exerts on the earth has the same magnitude as the force that the earth exerts on you. When you fall from a diving board into a swimming pool, the entire earth rises up to meet you! (You don't notice this because the earth's mass is greater than yours by a factor of about 10^{23}. Hence the earth's acceleration is only 10^{-23} as great as yours.)

Gravitation and Spherically Symmetric Bodies

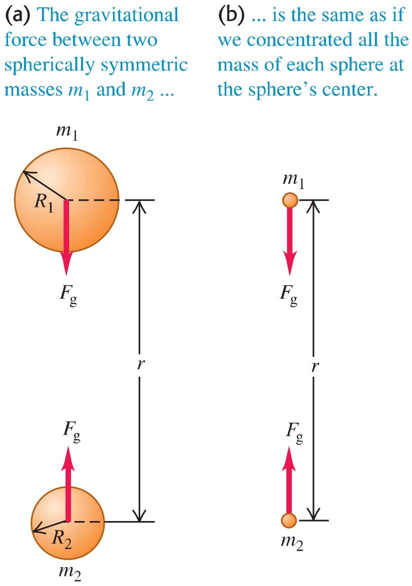
We have stated the law of gravitation in terms of the interaction between two *particles.* It turns out that the gravitational interaction of any two bodies having *spherically symmetric* mass distributions (such as solid spheres or spherical shells) is the same as though we concentrated all the mass of each at its center, as in Fig. 13.2. Thus, if we model the earth as a spherically symmetric body with mass m_E, the force it exerts on a particle or a spherically symmetric body with mass m, at a distance r between centers, is

$$F_g = \frac{Gm_Em}{r^2} \tag{13.2}$$

provided that the body lies outside the earth. A force of the same magnitude is exerted *on* the earth by the body. (We will prove these statements in Section 13.6.)

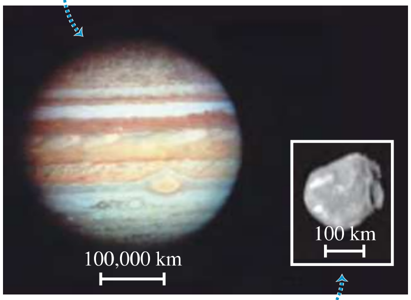
At points *inside* the earth the situation is different. If we could drill a hole to the center of the earth and measure the gravitational force on a body at various depths, we would find that toward the center of the earth the force *decreases,*

13.2 The gravitational effect *outside* any spherically symmetric mass distribution is the same as though all of the mass were concentrated at its center.



13.3 Spherical and nonspherical bodies: the planet Jupiter and one of Jupiter's small moons, Amalthea.

Jupiter's mass is very large (1.90×10^{27} kg), so the mutual gravitational attraction of its parts has pulled it into a nearly spherical shape.



Amalthea, one of Jupiter's small moons, has a relatively tiny mass (7.17×10^{18} kg, only about 3.8×10^{-9} the mass of Jupiter) and weak mutual gravitation, so it has an irregular shape.

rather than increasing as $1/r^2$. As the body enters the interior of the earth (or other spherical body), some of the earth's mass is on the side of the body opposite from the center and pulls in the opposite direction. Exactly at the center, the earth's gravitational force on the body is zero.

Spherically symmetric bodies are an important case because moons, planets, and stars all tend to be spherical. Since all particles in a body gravitationally attract each other, the particles tend to move to minimize the distance between them. As a result, the body naturally tends to assume a spherical shape, just as a lump of clay forms into a sphere if you squeeze it with equal forces on all sides. This effect is greatly reduced in celestial bodies of low mass, since the gravitational attraction is less, and these bodies tend *not* to be spherical (Fig. 13.3).

Determining the Value of G

To determine the value of the gravitational constant G, we have to *measure* the gravitational force between two bodies of known masses m_1 and m_2 at a known distance r. The force is extremely small for bodies that are small enough to be brought into the laboratory, but it can be measured with an instrument called a *torsion balance,* which Sir Henry Cavendish used in 1798 to determine G.

Figure 13.4 shows a modern version of the Cavendish torsion balance. A light, rigid rod shaped like an inverted T is supported by a very thin, vertical quartz fiber. Two small spheres, each of mass m_1, are mounted at the ends of the horizontal arms of the T. When we bring two large spheres, each of mass m_2, to the positions shown, the attractive gravitational forces twist the T through a small angle. To measure this angle, we shine a beam of light on a mirror fastened to the T. The reflected beam strikes a scale, and as the T twists, the reflected beam moves along the scale.

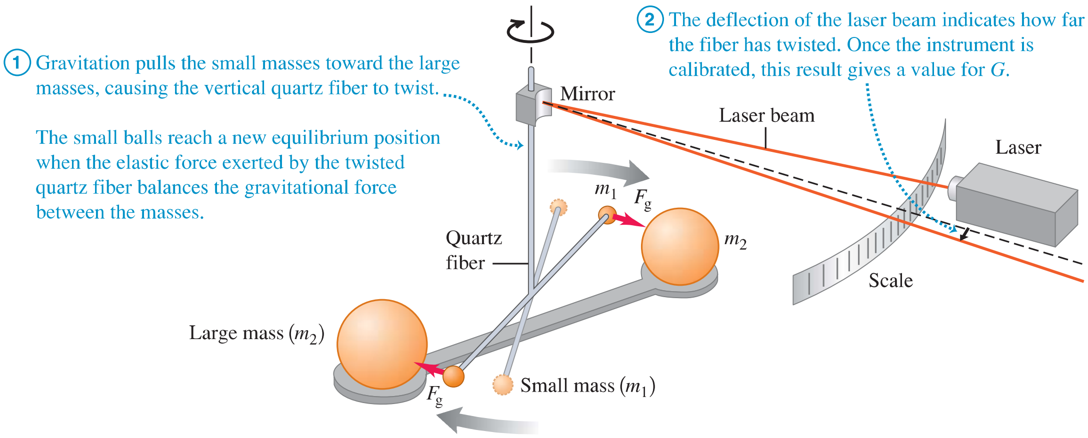
After calibrating the Cavendish balance, we can measure gravitational forces and thus determine G. The presently accepted value is

$$G = 6.67428(67) \times 10^{-11}\ \mathrm{N \cdot m^2/kg^2}$$

To three significant figures, $G = 6.67 \times 10^{-11}\ \mathrm{N \cdot m^2/kg^2}$. Because $1\ \mathrm{N} = 1\ \mathrm{kg \cdot m/s^2}$, the units of G can also be expressed as $\mathrm{m^3/(kg \cdot s^2)}$.

Gravitational forces combine vectorially. If each of two masses exerts a force on a third, the *total* force on the third mass is the vector sum of the individual forces of the first two. Example 13.3 makes use of this property, which is often called *superposition of forces.*

13.4 The principle of the Cavendish balance, used for determining the value of G. The angle of deflection has been exaggerated here for clarity.



Example 13.1 Calculating gravitational force

The mass m_1 of one of the small spheres of a Cavendish balance is 0.0100 kg, the mass m_2 of the nearest large sphere is 0.500 kg, and the center-to-center distance between them is 0.0500 m. Find the gravitational force F_g on each sphere due to the other.

SOLUTION

IDENTIFY, SET UP, and EXECUTE: Because the spheres are spherically symmetric, we can calculate F_g by treating them as *particles* separated by 0.0500 m, as in Fig. 13.2. Each sphere experiences the same magnitude of force from the other sphere. We use Newton's law of gravitation, Eq. (13.1), to determine F_g:

$$F_g = \frac{(6.67 \times 10^{-11}\ \text{N}\cdot\text{m}^2/\text{kg}^2)(0.0100\ \text{kg})(0.500\ \text{kg})}{(0.0500\ \text{m})^2}$$

$$= 1.33 \times 10^{-10}\ \text{N}$$

EVALUATE: It's remarkable that such a small force could be measured—or even detected—more than 200 years ago. Only a very massive object such as the earth exerts a gravitational force we can feel.

Example 13.2 Acceleration due to gravitational attraction

Suppose the two spheres in Example 13.1 are placed with their centers 0.0500 m apart at a point in space far removed from all other bodies. What is the magnitude of the acceleration of each, relative to an inertial system?

SOLUTION

IDENTIFY, SET UP, and EXECUTE: Each sphere exerts on the other a gravitational force of the same magnitude F_g, which we found in Example 13.1. We can neglect any other forces. The *acceleration* magnitudes a_1 and a_2 are different because the masses are different. To determine these we'll use Newton's second law:

$$a_1 = \frac{F_g}{m_1} = \frac{1.33 \times 10^{-10}\ \text{N}}{0.0100\ \text{kg}} = 1.33 \times 10^{-8}\ \text{m/s}^2$$

$$a_2 = \frac{F_g}{m_2} = \frac{1.33 \times 10^{-10}\ \text{N}}{0.500\ \text{kg}} = 2.66 \times 10^{-10}\ \text{m/s}^2$$

EVALUATE: The larger sphere has 50 times the mass of the smaller one and hence has $\frac{1}{50}$ the acceleration. These accelerations are *not* constant; the gravitational forces increase as the spheres move toward each other.

Example 13.3 Superposition of gravitational forces

Many stars belong to *systems* of two or more stars held together by their mutual gravitational attraction. Figure 13.5 shows a three-star system at an instant when the stars are at the vertices of a 45° right triangle. Find the total gravitational force exerted on the small star by the two large ones.

SOLUTION

IDENTIFY, SET UP, and EXECUTE: We use the principle of superposition: The total force $\vec{F}$ on the small star is the vector sum of the forces $\vec{F}_1$ and $\vec{F}_2$ due to each large star, as Fig. 13.5 shows. We assume that the stars are spheres as in Fig. 13.2. We first calculate the magnitudes F_1 and F_2 using Eq. (13.1) and then compute the vector sum using components:

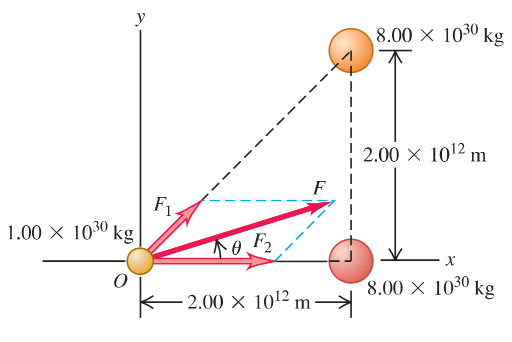
$$F_1 = \frac{\begin{bmatrix} (6.67 \times 10^{-11}\ \text{N}\cdot\text{m}^2/\text{kg}^2) \\ \times (8.00 \times 10^{30}\ \text{kg})(1.00 \times 10^{30}\ \text{kg}) \end{bmatrix}}{(2.00 \times 10^{12}\ \text{m})^2 + (2.00 \times 10^{12}\ \text{m})^2}$$

$$= 6.67 \times 10^{25}\ \text{N}$$

$$F_2 = \frac{\begin{bmatrix} (6.67 \times 10^{-11}\ \text{N}\cdot\text{m}^2/\text{kg}^2) \\ \times (8.00 \times 10^{30}\ \text{kg})(1.00 \times 10^{30}\ \text{kg}) \end{bmatrix}}{(2.00 \times 10^{12}\ \text{m})^2}$$

$$= 1.33 \times 10^{26}\ \text{N}$$

13.5 The total gravitational force on the small star (at O) is the vector sum of the forces exerted on it by the two larger stars. (For comparison, the mass of the sun—a rather ordinary star—is 1.99×10^{30} kg and the earth–sun distance is 1.50×10^{11} m.)



The x- and y-components of these forces are

$$F_{1x} = (6.67 \times 10^{25}\ \text{N})(\cos 45°) = 4.72 \times 10^{25}\ \text{N}$$

$$F_{1y} = (6.67 \times 10^{25}\ \text{N})(\sin 45°) = 4.72 \times 10^{25}\ \text{N}$$

$$F_{2x} = 1.33 \times 10^{26}\ \text{N}$$

$$F_{2y} = 0$$

Continued

The components of the total force $\vec{F}$ on the small star are

$$F_x = F_{1x} + F_{2x} = 1.81 \times 10^{26}\ \text{N}$$
$$F_y = F_{1y} + F_{2y} = 4.72 \times 10^{25}\ \text{N}$$

The magnitude of $\vec{F}$ and its angle θ (see Fig. 13.5) are

$$F = \sqrt{F_x{}^2 + F_y{}^2} = \sqrt{(1.81 \times 10^{26}\ \text{N})^2 + (4.72 \times 10^{25}\ \text{N})^2}$$
$$= 1.87 \times 10^{26}\ \text{N}$$
$$\theta = \arctan\frac{F_y}{F_x} = \arctan\frac{4.72 \times 10^{25}\ \text{N}}{1.81 \times 10^{26}\ \text{N}} = 14.6°$$

EVALUATE: While the force magnitude F is tremendous, the magnitude of the resulting acceleration is not: $a = F/m = (1.87 \times 10^{26}\ \text{N})/(1.00 \times 10^{30}\ \text{kg}) = 1.87 \times 10^{-4}\ \text{m/s}^2$. Furthermore, the force $\vec{F}$ is *not* directed toward the center of mass of the two large stars.

Why Gravitational Forces Are Important

13.6 Our solar system is part of a spiral galaxy like this one, which contains roughly 10^{11} stars as well as gas, dust, and other matter. The entire assemblage is held together by the mutual gravitational attraction of all the matter in the galaxy.

Comparing Examples 13.1 and 13.3 shows that gravitational forces are negligible between ordinary household-sized objects, but very substantial between objects that are the size of stars. Indeed, gravitation is *the* most important force on the scale of planets, stars, and galaxies (Fig. 13.6). It is responsible for holding our earth together and for keeping the planets in orbit about the sun. The mutual gravitational attraction between different parts of the sun compresses material at the sun's core to very high densities and temperatures, making it possible for nuclear reactions to take place there. These reactions generate the sun's energy output, which makes it possible for life to exist on earth and for you to read these words.

The gravitational force is so important on the cosmic scale because it acts *at a distance,* without any direct contact between bodies. Electric and magnetic forces have this same remarkable property, but they are less important on astronomical scales because large accumulations of matter are electrically neutral; that is, they contain equal amounts of positive and negative charge. As a result, the electric and magnetic forces between stars or planets are very small or zero. The strong and weak interactions that we discussed in Section 5.5 also act at a distance, but their influence is negligible at distances much greater than the diameter of an atomic nucleus (about 10^{-14} m).

A useful way to describe forces that act at a distance is in terms of a *field.* One body sets up a disturbance or field at all points in space, and the force that acts on a second body at a particular point is its response to the first body's field at that point. There is a field associated with each force that acts at a distance, and so we refer to gravitational fields, electric fields, magnetic fields, and so on. We won't need the field concept for our study of gravitation in this chapter, so we won't discuss it further here. But in later chapters we'll find that the field concept is an extraordinarily powerful tool for describing electric and magnetic interactions.

Test Your Understanding of Section 13.1 The planet Saturn has about 100 times the mass of the earth and is about 10 times farther from the sun than the earth is. Compared to the acceleration of the earth caused by the sun's gravitational pull, how great is the acceleration of Saturn due to the sun's gravitation? (i) 100 times greater; (ii) 10 times greater; (iii) the same; (iv) $\frac{1}{10}$ as great; (v) $\frac{1}{100}$ as great. ❙

MasteringPHYSICS

PhET: Lunar Lander

13.2 Weight

We defined the *weight* of a body in Section 4.4 as the attractive gravitational force exerted on it by the earth. We can now broaden our definition:

The weight of a body is the total gravitational force exerted on the body by all other bodies in the universe.

When the body is near the surface of the earth, we can neglect all other gravitational forces and consider the weight as just the earth's gravitational attraction. At the surface of the *moon* we consider a body's weight to be the gravitational attraction of the moon, and so on.

If we again model the earth as a spherically symmetric body with radius R_E and mass m_E, the weight w of a small body of mass m at the earth's surface (a distance R_E from its center) is

$$w = F_g = \frac{Gm_E m}{R_E^2} \quad \text{(weight of a body of mass } m \text{ at the earth's surface)} \tag{13.3}$$

But we also know from Section 4.4 that the weight w of a body is the force that causes the acceleration g of free fall, so by Newton's second law, $w = mg$. Equating this with Eq. (13.3) and dividing by m, we find

$$g = \frac{Gm_E}{R_E^2} \quad \text{(acceleration due to gravity at the earth's surface)} \tag{13.4}$$

The acceleration due to gravity g is independent of the mass m of the body because m doesn't appear in this equation. We already knew that, but we can now see how it follows from the law of gravitation.

We can *measure* all the quantities in Eq. (13.4) except for m_E, so this relationship allows us to compute the mass of the earth. Solving Eq. (13.4) for m_E and using $R_E = 6380 \text{ km} = 6.38 \times 10^6 \text{ m}$ and $g = 9.80 \text{ m/s}^2$, we find

$$m_E = \frac{gR_E^2}{G} = 5.98 \times 10^{24} \text{ kg}$$

This is very close to the currently accepted value of 5.974×10^{24} kg. Once Cavendish had measured G, he computed the mass of the earth in just this way.

At a point above the earth's surface a distance r from the center of the earth (a distance $r - R_E$ above the surface), the weight of a body is given by Eq. (13.3) with R_E replaced by r:

$$w = F_g = \frac{Gm_E m}{r^2} \tag{13.5}$$

The weight of a body decreases inversely with the square of its distance from the earth's center (Fig. 13.7). Figure 13.8 shows how the weight varies with height above the earth for an astronaut who weighs 700 N at the earth's surface.

The *apparent* weight of a body on earth differs slightly from the earth's gravitational force because the earth rotates and is therefore not precisely an inertial frame of reference. We have ignored this effect in our earlier discussion and have assumed that the earth *is* an inertial system. We will return to the effect of the earth's rotation in Section 13.7.

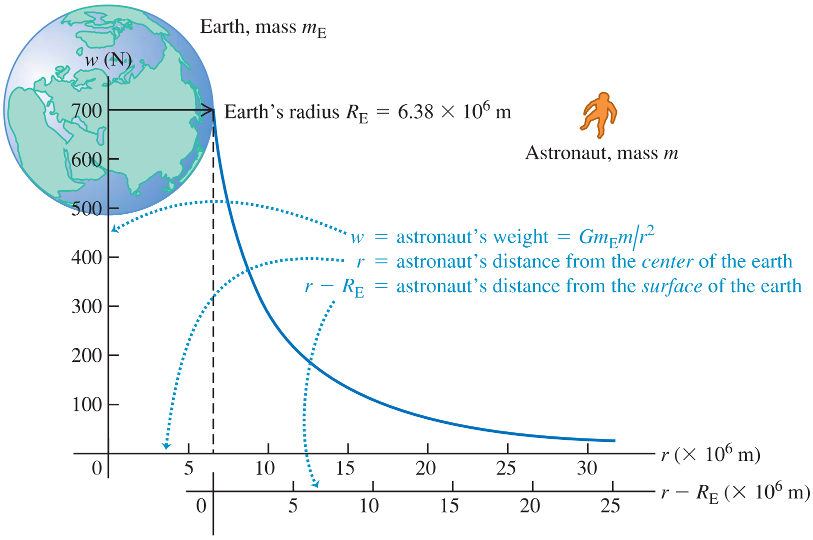
While the earth is an approximately spherically symmetric distribution of mass, it is *not* uniform throughout its volume. To demonstrate this, let's first calculate the average *density,* or mass per unit volume, of the earth. If we assume a spherical earth, the volume is

$$V_E = \tfrac{4}{3}\pi R_E^3 = \tfrac{4}{3}\pi(6.38 \times 10^6 \text{ m})^3 = 1.09 \times 10^{21} \text{ m}^3$$

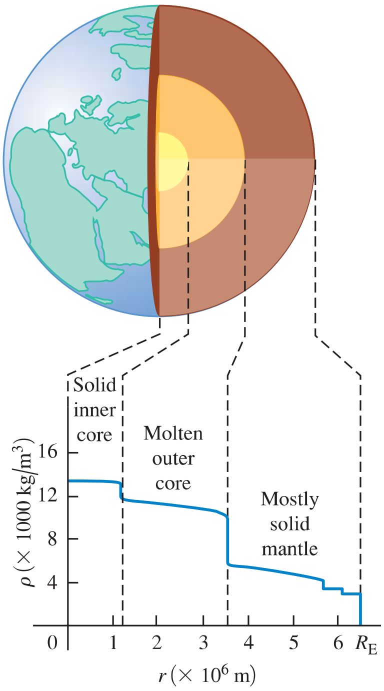
Application Walking and Running on the Moon
You automatically transition from a walk to a run when the vertical force you exert on the ground—which, by Newton's third law, equals the vertical force the ground exerts on you—exceeds your weight. This transition from walking to running happens at much lower speeds on the moon, where objects weigh only 17% as much as on earth. Hence, the Apollo astronauts found themselves running even when moving relatively slowly during their moon "walks."

13.7 In an airliner at high altitude, you are farther from the center of the earth than when on the ground and hence weigh slightly less. Can you show that at an altitude of 10 km above the surface, you weigh 0.3% less than you do on the ground?

13.8 An astronaut who weighs 700 N at the earth's surface experiences less gravitational attraction when above the surface. The relevant distance r is from the astronaut to the *center* of the earth (*not* from the astronaut to the earth's surface).



13.9 The density of the earth decreases with increasing distance from its center.



The average density ρ (the Greek letter rho) of the earth is the total mass divided by the total volume:

$$\rho = \frac{m_E}{V_E} = \frac{5.97 \times 10^{24}\ \text{kg}}{1.09 \times 10^{21}\ \text{m}^3}$$
$$= 5500\ \text{kg/m}^3 = 5.5\ \text{g/cm}^3$$

(For comparison, the density of water is $1000\ \text{kg/m}^3 = 1.00\ \text{g/cm}^3$.) If the earth were uniform, we would expect rocks near the earth's surface to have this same density. In fact, the density of surface rocks is substantially lower, ranging from about $2000\ \text{kg/m}^3$ for sedimentary rocks to about $3300\ \text{kg/m}^3$ for basalt. So the earth *cannot* be uniform, and the interior of the earth must be much more dense than the surface in order that the *average* density be $5500\ \text{kg/m}^3$. According to geophysical models of the earth's interior, the maximum density at the center is about $13{,}000\ \text{kg/m}^3$. Figure 13.9 is a graph of density as a function of distance from the center.

Example 13.4 Gravity on Mars

A robotic lander with an earth weight of 3430 N is sent to Mars, which has radius $R_M = 3.40 \times 10^6$ m and mass $m_M = 6.42 \times 10^{23}$ kg (see Appendix F). Find the weight F_g of the lander on the Martian surface and the acceleration there due to gravity, g_M.

SOLUTION

IDENTIFY and SET UP: To find F_g we use Eq. (13.3), replacing m_E and R_E with m_M and R_M. We determine the lander mass m from the lander's earth weight w and then find g_M from $F_g = mg_M$.

EXECUTE: The lander's earth weight is $w = mg$, so

$$m = \frac{w}{g} = \frac{3430\ \text{N}}{9.80\ \text{m/s}^2} = 350\ \text{kg}$$

The mass is the same no matter where the lander is. From Eq. (13.3), the lander's weight on Mars is

$$F_g = \frac{Gm_Mm}{R_M^2}$$
$$= \frac{(6.67 \times 10^{-11}\ \text{N} \cdot \text{m}^2/\text{kg}^2)(6.42 \times 10^{23}\ \text{kg})(350\ \text{kg})}{(3.40 \times 10^6\ \text{m})^2}$$
$$= 1.30 \times 10^3\ \text{N}$$

The acceleration due to gravity on Mars is

$$g_M = \frac{F_g}{m} = \frac{1.30 \times 10^3 \text{ N}}{350 \text{ kg}} = 3.7 \text{ m/s}^2$$

EVALUATE: Even though Mars has just 11% of the earth's mass (6.42×10^{23} kg versus 5.98×10^{24} kg), the acceleration due to gravity g_M (and hence an object's weight F_g) is roughly 40% as large as on earth. That's because g_M is also inversely proportional to the square of the planet's radius, and Mars has only 53% the radius of earth (3.40×10^6 m versus 6.38×10^6 m).

You can check our result for g_M by using Eq. (13.4), with appropriate replacements. Do you get the same answer?

Test Your Understanding of Section 13.2 Rank the following hypothetical planets in order from highest to lowest value of g at the surface:
(i) mass = 2 times the mass of the earth, radius = 2 times the radius of the earth;
(ii) mass = 4 times the mass of the earth, radius = 4 times the radius of the earth;
(iii) mass = 4 times the mass of the earth, radius = 2 times the radius of the earth;
(iv) mass = 2 times the mass of the earth, radius = 4 times the radius of the earth.

13.3 Gravitational Potential Energy

When we first introduced gravitational potential energy in Section 7.1, we assumed that the gravitational force on a body is constant in magnitude and direction. This led to the expression $U = mgy$. But the earth's gravitational force on a body of mass m at any point outside the earth is given more generally by Eq. (13.2), $F_g = Gm_Em/r^2$, where m_E is the mass of the earth and r is the distance of the body from the earth's center. For problems in which r changes enough that the gravitational force can't be considered constant, we need a more general expression for gravitational potential energy.

To find this expression, we follow the same steps as in Section 7.1. We consider a body of mass m outside the earth, and first compute the work W_{grav} done by the gravitational force when the body moves directly away from or toward the center of the earth from $r = r_1$ to $r = r_2$, as in Fig. 13.10. This work is given by

$$W_{grav} = \int_{r_1}^{r_2} F_r\, dr \tag{13.6}$$

where F_r is the radial component of the gravitational force $\vec{F}$—that is, the component in the direction *outward* from the center of the earth. Because $\vec{F}$ points directly *inward* toward the center of the earth, F_r is negative. It differs from Eq. (13.2), the magnitude of the gravitational force, by a minus sign:

$$F_r = -\frac{Gm_Em}{r^2} \tag{13.7}$$

Substituting Eq. (13.7) into Eq. (13.6), we see that W_{grav} is given by

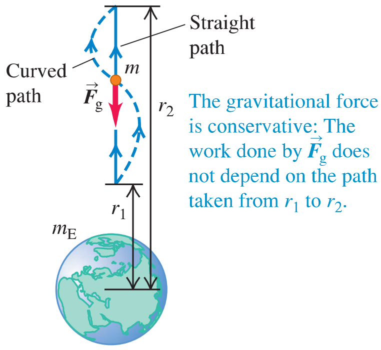
$$W_{grav} = -Gm_Em\int_{r_1}^{r_2} \frac{dr}{r^2} = \frac{Gm_Em}{r_2} - \frac{Gm_Em}{r_1} \tag{13.8}$$

The path doesn't have to be a straight line; it could also be a curve like the one in Fig. 13.10. By an argument similar to that in Section 7.1, this work depends only on the initial and final values of r, not on the path taken. This also proves that the gravitational force is always *conservative.*

We now define the corresponding potential energy U so that $W_{grav} = U_1 - U_2$, as in Eq. (7.3). Comparing this with Eq. (13.8), we see that the appropriate definition for **gravitational potential energy** is

$$U = -\frac{Gm_Em}{r} \quad \text{(gravitational potential energy)} \tag{13.9}$$

13.10 Calculating the work done on a body by the gravitational force as the body moves from radial coordinate r_1 to r_2.



13.11 A graph of the gravitational potential energy U for the system of the earth (mass m_E) and an astronaut (mass m) versus the astronaut's distance r from the center of the earth.

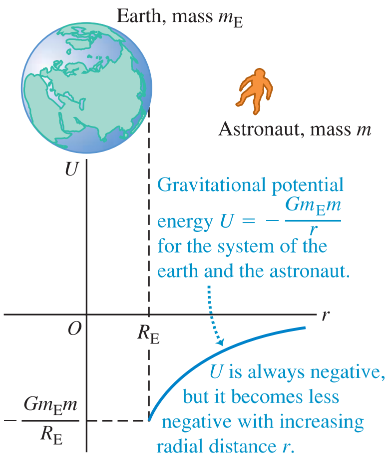


Figure 13.11 shows how the gravitational potential energy depends on the distance r between the body of mass m and the center of the earth. When the body moves away from the earth, r increases, the gravitational force does negative work, and U increases (i.e., becomes less negative). When the body "falls" toward earth, r decreases, the gravitational work is positive, and the potential energy decreases (i.e., becomes more negative).

You may be troubled by Eq. (13.9) because it states that gravitational potential energy is always negative. But in fact you've seen negative values of U before. In using the formula $U = mgy$ in Section 7.1, we found that U was negative whenever the body of mass m was at a value of y below the arbitrary height we chose to be $y = 0$—that is, whenever the body and the earth were closer together than some certain arbitrary distance. (See, for instance, Example 7.2 in Section 7.1.) In defining U by Eq. (13.9), we have chosen U to be zero when the body of mass m is infinitely far from the earth $(r = \infty)$. As the body moves toward the earth, gravitational potential energy decreases and so becomes negative.

If we wanted, we could make $U = 0$ at the surface of the earth, where $r = R_E$, by simply adding the quantity $Gm_E m/R_E$ to Eq. (13.9). This would make U positive when $r > R_E$. We won't do this for two reasons: One, it would make the expression for U more complicated; and two, the added term would not affect the *difference* in potential energy between any two points, which is the only physically significant quantity.

CAUTION **Gravitational force vs. gravitational potential energy** Be careful not to confuse the expressions for gravitational force, Eq. (13.7), and gravitational potential energy, Eq. (13.9). The force F_r is proportional to $1/r^2$, while potential energy U is proportional to $1/r$.

Armed with Eq. (13.9), we can now use general energy relationships for problems in which the $1/r^2$ behavior of the earth's gravitational force has to be included. If the gravitational force on the body is the only force that does work, the total mechanical energy of the system is constant, or *conserved.* In the following example we'll use this principle to calculate **escape speed,** the speed required for a body to escape completely from a planet.

Example 13.5 "From the earth to the moon"

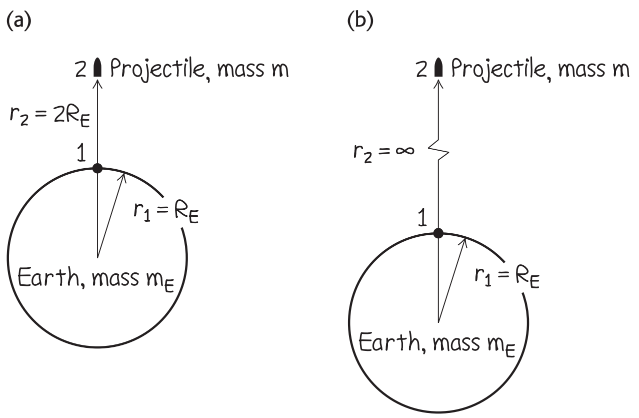
In Jules Verne's 1865 story with this title, three men went to the moon in a shell fired from a giant cannon sunk in the earth in Florida. (a) Find the minimum muzzle speed needed to shoot a shell straight up to a height above the earth equal to the earth's radius R_E. (b) Find the minimum muzzle speed that would allow a shell to escape from the earth completely (the *escape speed*). Neglect air resistance, the earth's rotation, and the gravitational pull of the moon. The earth's radius and mass are $R_E = 6.38 \times 10^6$ m and $m_E = 5.97 \times 10^{24}$ kg.

SOLUTION

IDENTIFY and SET UP: Once the shell leaves the cannon muzzle, only the (conservative) gravitational force does work. Hence we can use conservation of mechanical energy to find the speed at which the shell must leave the muzzle so as to come to a halt (a) at two earth radii from the earth's center and (b) at an infinite distance from earth. The energy-conservation equation is $K_1 + U_1 = K_2 + U_2$, with U given by Eq. (13.9).

Figure 13.12 shows our sketches. Point 1 is at $r_1 = R_E$, where the shell leaves the cannon with speed v_1 (the target variable). Point 2 is where the shell reaches its maximum height; in part (a) $r_2 = 2R_E$ (Fig. 13.12a), and in part (b) $r_2 = \infty$ (Fig 13.12b). In both cases $v_2 = 0$ and $K_2 = 0$. Let m be the mass of the shell (with passengers).

13.12 Our sketches for this problem.



EXECUTE: (a) We solve the energy-conservation equation for v_1:

$$K_1 + U_1 = K_2 + U_2$$

$$\frac{1}{2}mv_1^2 + \left(-\frac{Gm_Em}{R_E}\right) = 0 + \left(-\frac{Gm_Em}{2R_E}\right)$$

$$v_1 = \sqrt{\frac{Gm_E}{R_E}} = \sqrt{\frac{(6.67 \times 10^{-11}\ \mathrm{N \cdot m^2/kg^2})(5.97 \times 10^{24}\ \mathrm{kg})}{6.38 \times 10^6\ \mathrm{m}}}$$

$$= 7900\ \mathrm{m/s}\ (= 28{,}400\ \mathrm{km/h} = 17{,}700\ \mathrm{mi/h})$$

(b) Now $r_2 = \infty$ so $U_2 = 0$ (see Fig. 13.11). Since $K_2 = 0$, the total mechanical energy $K_2 + U_2$ is zero in this case. Again we solve the energy-conservation equation for v_1:

$$\frac{1}{2}mv_1^2 + \left(-\frac{Gm_Em}{R_E}\right) = 0 + 0$$

$$v_1 = \sqrt{\frac{2Gm_E}{R_E}}$$

$$= \sqrt{\frac{2(6.67 \times 10^{-11}\ \mathrm{N \cdot m^2/kg^2})(5.97 \times 10^{24}\ \mathrm{kg})}{6.38 \times 10^6\ \mathrm{m}}}$$

$$= 1.12 \times 10^4\ \mathrm{m/s}\ (= 40{,}200\ \mathrm{km/h} = 25{,}000\ \mathrm{mi/h})$$

EVALUATE: Our result in part (b) doesn't depend on the mass of the shell or the direction of launch. A modern spacecraft launched from Florida must attain essentially the speed found in part (b) to escape the earth; however, before launch it's already moving at 410 m/s to the east because of the earth's rotation. Launching to the east takes advantage of this "free" contribution toward escape speed.

To generalize, the initial speed v_1 needed for a body to escape from the surface of a spherical body of mass M and radius R (ignoring air resistance) is $v_1 = \sqrt{2GM/R}$ (escape speed). This equation yields escape speeds of 5.02×10^3 m/s for Mars, 5.95×10^4 m/s for Jupiter, and 6.18×10^5 m/s for the sun.

More on Gravitational Potential Energy

As a final note, let's show that when we are close to the earth's surface, Eq. (13.9) reduces to the familiar $U = mgy$ from Chapter 7. We first rewrite Eq. (13.8) as

$$W_{\mathrm{grav}} = Gm_Em\frac{r_1 - r_2}{r_1 r_2}$$

If the body stays close to the earth, then in the denominator we may replace r_1 and r_2 by R_E, the earth's radius, so

$$W_{\mathrm{grav}} = Gm_Em\frac{r_1 - r_2}{R_E^2}$$

According to Eq. (13.4), $g = Gm_E/R_E^2$, so

$$W_{\mathrm{grav}} = mg(r_1 - r_2)$$

If we replace the r's by y's, this is just Eq. (7.1) for the work done by a constant gravitational force. In Section 7.1 we used this equation to derive Eq. (7.2), $U = mgy$, so we may consider Eq. (7.2) for gravitational potential energy to be a special case of the more general Eq. (13.9).

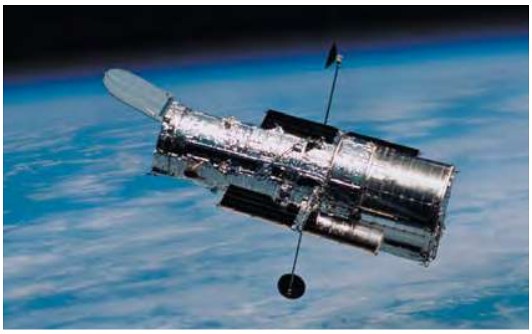
Test Your Understanding of Section 13.3 Is it possible for a planet to have the same surface gravity as the earth (that is, the same value of g at the surface) and yet have a greater escape speed?

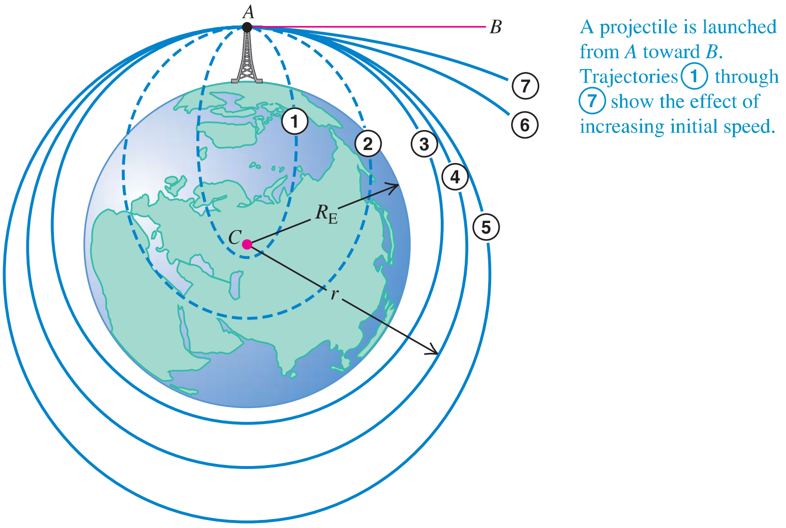
13.4 The Motion of Satellites

Artificial satellites orbiting the earth are a familiar part of modern technology (Fig. 13.13). But how do they stay in orbit, and what determines the properties of their orbits? We can use Newton's laws and the law of gravitation to provide the answers. We'll see in the next section that the motion of planets can be analyzed in the same way.

To begin, think back to the discussion of projectile motion in Section 3.3. In Example 3.6 a motorcycle rider rides horizontally off the edge of a cliff, launching himself into a parabolic path that ends on the flat ground at the base of the cliff. If he survives and repeats the experiment with increased launch speed, he will land farther from the starting point. We can imagine him launching himself with great enough speed that the earth's curvature becomes significant. As he falls, the earth curves away beneath him. If he is going fast enough, and if his

13.13 With a length of 13.2 m and a mass of 11,000 kg, the Hubble Space Telescope is among the largest satellites placed in orbit.





13.14 Trajectories of a projectile launched from a great height (ignoring air resistance). Orbits 1 and 2 would be completed as shown if the earth were a point mass at C. (This illustration is based on one in Isaac Newton's *Principia.*)

launch point is high enough that he clears the mountaintops, he may be able to go right on around the earth without ever landing.

Figure 13.14 shows a variation on this theme. We launch a projectile from point A in the direction AB, tangent to the earth's surface. Trajectories 1 through 7 show the effect of increasing the initial speed. In trajectories 3 through 5 the projectile misses the earth and becomes a satellite. If there is no retarding force, the projectile's speed when it returns to point A is the same as its initial speed and it repeats its motion indefinitely.

Trajectories 1 through 5 close on themselves and are called **closed orbits.** All closed orbits are ellipses or segments of ellipses; trajectory 4 is a circle, a special case of an ellipse. (We'll discuss the properties of an ellipse in Section 13.5.) Trajectories 6 and 7 are **open orbits.** For these paths the projectile never returns to its starting point but travels ever farther away from the earth.

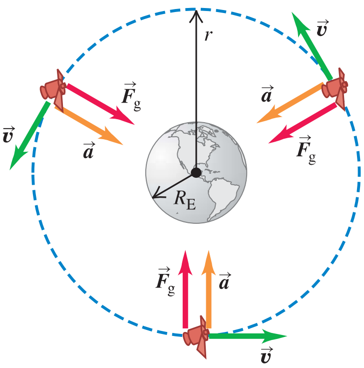
MasteringPHYSICS

PhET: My Solar System
ActivPhysics 4.6: Satellites Orbit

Satellites: Circular Orbits

A *circular* orbit, like trajectory 4 in Fig. 13.14, is the simplest case. It is also an important case, since many artificial satellites have nearly circular orbits and the orbits of the planets around the sun are also fairly circular. The only force acting on a satellite in circular orbit around the earth is the earth's gravitational attraction, which is directed toward the center of the earth and hence toward the center of the orbit (Fig. 13.15). As we discussed in Section 5.4, this means that the satellite is in *uniform* circular motion and its speed is constant. The satellite isn't falling *toward* the earth; rather, it's constantly falling *around* the earth. In a circular orbit the speed is just right to keep the distance from the satellite to the center of the earth constant.

13.15 The force $\vec{F}_g$ due to the earth's gravitational attraction provides the centripetal acceleration that keeps a satellite in orbit. Compare to Fig. 5.28.



The satellite is in a circular orbit: Its acceleration $\vec{a}$ is always perpendicular to its velocity $\vec{v}$, so its speed v is constant.

Let's see how to find the constant speed v of a satellite in a circular orbit. ? The radius of the orbit is r, measured from the *center* of the earth; the acceleration of the satellite has magnitude $a_{rad} = v^2/r$ and is always directed toward the center of the circle. By the law of gravitation, the net force (gravitational force) on the satellite of mass m has magnitude $F_g = Gm_E m/r^2$ and is in the same direction as the acceleration. Newton's second law $(\sum\vec{F} = m\vec{a})$ then tells us that

$$\frac{Gm_E m}{r^2} = \frac{mv^2}{r}$$

Solving this for v, we find

$$v = \sqrt{\frac{Gm_E}{r}} \quad \text{(circular orbit)} \qquad (13.10)$$

This relationship shows that we can't choose the orbit radius r and the speed v independently; for a given radius r, the speed v for a circular orbit is determined.

The satellite's mass m doesn't appear in Eq. (13.10), which shows that the motion of a satellite does not depend on its mass. If we could cut a satellite in half without changing its speed, each half would continue on with the original motion. An astronaut on board a space shuttle is herself a satellite of the earth, held by the earth's gravitational attraction in the same orbit as the shuttle. The astronaut has the same velocity and acceleration as the shuttle, so nothing is pushing her against the floor or walls of the shuttle. She is in a state of *apparent weightlessness,* as in a freely falling elevator; see the discussion following Example 5.9 in Section 5.2. (*True* weightlessness would occur only if the astronaut were infinitely far from any other masses, so that the gravitational force on her would be zero.) Indeed, every part of her body is apparently weightless; she feels nothing pushing her stomach against her intestines or her head against her shoulders (Fig. 13.16).

13.16 These space shuttle astronauts are in a state of apparent weightlessness. Which are right side up and which are upside down?

Apparent weightlessness is not just a feature of circular orbits; it occurs whenever gravity is the only force acting on a spacecraft. Hence it occurs for orbits of any shape, including open orbits such as trajectories 6 and 7 in Fig. 13.14.

We can derive a relationship between the radius r of a circular orbit and the period T, the time for one revolution. The speed v is the distance $2\pi r$ traveled in one revolution, divided by the period:

$$v = \frac{2\pi r}{T} \tag{13.11}$$

To get an expression for T, we solve Eq. (13.11) for T and substitute v from Eq. (13.10):

$$T = \frac{2\pi r}{v} = 2\pi r\sqrt{\frac{r}{Gm_{\mathrm{E}}}} = \frac{2\pi r^{3/2}}{\sqrt{Gm_{\mathrm{E}}}} \quad \text{(circular orbit)} \tag{13.12}$$

Equations (13.10) and (13.12) show that larger orbits correspond to slower speeds and longer periods. As an example, the International Space Station orbits 6800 km from the center of the earth (400 km above the earth's surface) with an orbital speed of 7.7 km/s and an orbital period of 93 minutes. The moon orbits the earth in a much larger orbit of radius 384,000 km, and so has a much slower orbital speed (1.0 km/s) and a much longer orbital period (27.3 days).

It's interesting to compare Eq. (13.10) to the calculation of escape speed in Example 13.5. We see that the escape speed from a spherical body with radius R is $\sqrt{2}$ times greater than the speed of a satellite in a circular orbit at that radius. If our spacecraft is in circular orbit around *any* planet, we have to multiply our speed by a factor of $\sqrt{2}$ to escape to infinity, regardless of the planet's mass.

Since the speed v in a circular orbit is determined by Eq. (13.10) for a given orbit radius r, the total mechanical energy $E = K + U$ is determined as well. Using Eqs. (13.9) and (13.10), we have

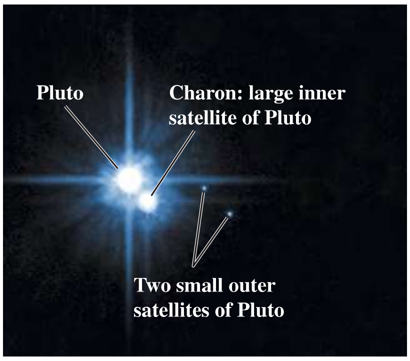
$$E = K + U = \tfrac{1}{2}mv^2 + \left(-\frac{Gm_{\mathrm{E}}m}{r}\right) = \tfrac{1}{2}m\left(\frac{Gm_{\mathrm{E}}}{r}\right) - \frac{Gm_{\mathrm{E}}m}{r}$$

$$= -\frac{Gm_{\mathrm{E}}m}{2r} \quad \text{(circular orbit)} \tag{13.13}$$

The total mechanical energy in a circular orbit is negative and equal to one-half the potential energy. Increasing the orbit radius r means increasing the mechanical energy (that is, making E less negative). If the satellite is in a relatively low orbit that encounters the outer fringes of earth's atmosphere, mechanical energy decreases due to negative work done by the force of air resistance; as a result, the orbit radius decreases until the satellite hits the ground or burns up in the atmosphere.

We have talked mostly about earth satellites, but we can apply the same analysis to the circular motion of *any* body under its gravitational attraction to a stationary body. Other examples include the earth's moon and the moons of other worlds (Fig. 13.17).

13.17 The two small satellites of the minor planet Pluto were discovered in 2005. In accordance with Eqs. (13.10) and (13.12), the satellite in the larger orbit has a slower orbital speed and a longer orbital period than the satellite in the smaller orbit.



Example 13.6 A satellite orbit

You wish to put a 1000-kg satellite into a circular orbit 300 km above the earth's surface. (a) What speed, period, and radial acceleration will it have? (b) How much work must be done to the satellite to put it in orbit? (c) How much additional work would have to be done to make the satellite escape the earth? The earth's radius and mass are given in Example 13.5 (Section 13.3).

SOLUTION

IDENTIFY and SET UP: The satellite is in a circular orbit, so we can use the equations derived in this section. In part (a), we first find the radius r of the satellite's orbit from its altitude. We then calculate the speed v and period T using Eqs. (13.10) and (13.12) and the acceleration from $a_{\mathrm{rad}} = v^2/r$. In parts (b) and (c), the work required is the difference between the initial and final mechanical energy, which for a circular orbit is given by Eq. (13.13).

EXECUTE: (a) The radius of the satellite's orbit is $r = 6380\ \mathrm{km} + 300\ \mathrm{km} = 6680\ \mathrm{km} = 6.68 \times 10^6\ \mathrm{m}$. From Eq. (13.10), the orbital speed is

$$v = \sqrt{\frac{Gm_{\mathrm{E}}}{r}} = \sqrt{\frac{(6.67 \times 10^{-11}\ \mathrm{N \cdot m^2/kg^2})(5.97 \times 10^{24}\ \mathrm{kg})}{6.68 \times 10^6\ \mathrm{m}}}$$
$$= 7720\ \mathrm{m/s}$$

We find the orbital period from Eq. (13.12):

$$T = \frac{2\pi r}{v} = \frac{2\pi(6.68 \times 10^6\ \mathrm{m})}{7720\ \mathrm{m/s}} = 5440\ \mathrm{s} = 90.6\ \mathrm{min}$$

Finally, the radial acceleration is

$$a_{\mathrm{rad}} = \frac{v^2}{r} = \frac{(7720\ \mathrm{m/s})^2}{6.68 \times 10^6\ \mathrm{m}} = 8.92\ \mathrm{m/s^2}$$

This is the value of g at a height of 300 km above the earth's surface; it is about 10% less than the value of g at the surface.

(b) The work required is the difference between E_2, the total mechanical energy when the satellite is in orbit, and E_1, the total mechanical energy when the satellite was at rest on the launch pad. From Eq. (13.13), the energy in orbit is

$$E_2 = -\frac{Gm_{\mathrm{E}}m}{2r}$$
$$= -\frac{(6.67 \times 10^{-11}\ \mathrm{N \cdot m^2/kg^2})(5.97 \times 10^{24}\ \mathrm{kg})(1000\ \mathrm{kg})}{2(6.68 \times 10^6\ \mathrm{m})}$$
$$= -2.98 \times 10^{10}\ \mathrm{J}$$

The satellite's kinetic energy is zero on the launch pad ($r = R_{\mathrm{E}}$), so

$$E_1 = K_1 + U_1 = 0 + \left(-\frac{Gm_{\mathrm{E}}m}{R_{\mathrm{E}}}\right)$$
$$= -\frac{(6.67 \times 10^{-11}\ \mathrm{N \cdot m^2/kg^2})(5.97 \times 10^{24}\ \mathrm{kg})(1000\ \mathrm{kg})}{6.38 \times 10^6\ \mathrm{m}}$$
$$= -6.24 \times 10^{10}\ \mathrm{J}$$

Hence the work required is

$$W_{\mathrm{required}} = E_2 - E_1 = (-2.98 \times 10^{10}\ \mathrm{J}) - (-6.24 \times 10^{10}\ \mathrm{J})$$
$$= 3.26 \times 10^{10}\ \mathrm{J}$$

(c) We saw in part (b) of Example 13.5 that the minimum total mechanical energy for a satellite to escape to infinity is zero. Here, the total mechanical energy in the circular orbit is $E_2 = -2.98 \times 10^{10}\ \mathrm{J}$; to increase this to zero, an amount of work equal to $2.98 \times 10^{10}\ \mathrm{J}$ would have to be done on the satellite, presumably by rocket engines attached to it.

EVALUATE: In part (b) we ignored the satellite's initial kinetic energy (while it was still on the launch pad) due to the rotation of the earth. How much difference does this make? (See Example 13.5 for useful data.)

Test Your Understanding of Section 13.4 Your personal spacecraft is in a low-altitude circular orbit around the earth. Air resistance from the outer regions of the atmosphere does negative work on the spacecraft, causing the orbital radius to decrease slightly. Does the speed of the spacecraft (i) remain the same, (ii) increase, or (iii) decrease?

13.5 Kepler's Laws and the Motion of Planets

The name *planet* comes from a Greek word meaning "wanderer," and indeed the planets continuously change their positions in the sky relative to the background of stars. One of the great intellectual accomplishments of the 16th and 17th centuries was the threefold realization that the earth is also a planet, that all planets orbit the sun, and that the apparent motions of the planets as seen from the earth can be used to precisely determine their orbits.

The first and second of these ideas were published by Nicolaus Copernicus in Poland in 1543. The nature of planetary orbits was deduced between 1601 and 1619 by the German astronomer and mathematician Johannes Kepler, using a voluminous set of precise data on apparent planetary motions compiled by his mentor, the Danish astronomer Tycho Brahe. By trial and error, Kepler

discovered three empirical laws that accurately described the motions of the planets:

1. **Each planet moves in an elliptical orbit, with the sun at one focus of the ellipse.**
2. **A line from the sun to a given planet sweeps out equal areas in equal times.**
3. **The periods of the planets are proportional to the $\frac{3}{2}$ powers of the major axis lengths of their orbits.**

Kepler did not know *why* the planets moved in this way. Three generations later, when Newton turned his attention to the motion of the planets, he discovered that each of Kepler's laws can be *derived;* they are consequences of Newton's laws of motion and the law of gravitation. Let's see how each of Kepler's laws arises.

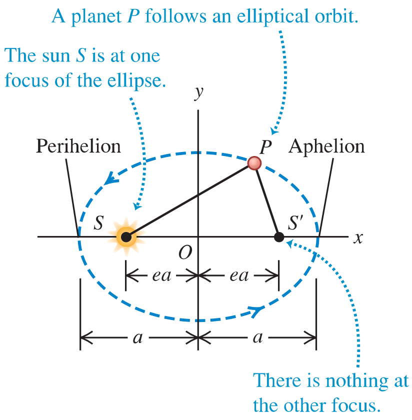
Kepler's First Law

First consider the elliptical orbits described in Kepler's first law. Figure 13.18 shows the geometry of an ellipse. The longest dimension is the *major axis,* with half-length a; this half-length is called the **semi-major axis.** The sum of the distances from S to P and from S' to P is the same for all points on the curve. S and S' are the *foci* (plural of *focus*). The sun is at S, and the planet is at P; we think of them both as points because the size of each is very small in comparison to the distance between them. There is nothing at the other focus S'.

The distance of each focus from the center of the ellipse is ea, where e is a dimensionless number between 0 and 1 called the **eccentricity.** If $e = 0$, the ellipse is a circle. The actual orbits of the planets are fairly circular; their eccentricities range from 0.007 for Venus to 0.206 for Mercury. (The earth's orbit has $e = 0.017$.) The point in the planet's orbit closest to the sun is the *perihelion,* and the point most distant from the sun is the *aphelion.*

Newton was able to show that for a body acted on by an attractive force proportional to $1/r^2$, the only possible closed orbits are a circle or an ellipse; he also showed that open orbits (trajectories 6 and 7 in Fig. 13.14) must be parabolas or hyperbolas. These results can be derived by a straightforward application of Newton's laws and the law of gravitation, together with a lot more differential equations than we're ready for.

13.18 Geometry of an ellipse. The sum of the distances SP and $S'P$ is the same for every point on the curve. The sizes of the sun (S) and planet (P) are exaggerated for clarity.



Kepler's Second Law

Figure 13.19 shows Kepler's second law. In a small time interval dt, the line from the sun S to the planet P turns through an angle $d\theta$. The area swept out is the colored triangle with height r, base length $r\,d\theta$, and area $dA = \frac{1}{2}r^2\,d\theta$ in Fig. 13.19b. The rate at which area is swept out, dA/dt, is called the *sector velocity:*

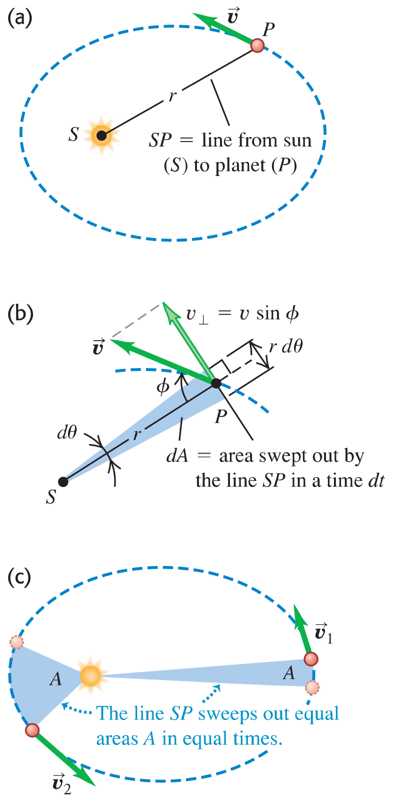
$$\frac{dA}{dt} = \frac{1}{2}r^2\frac{d\theta}{dt} \qquad (13.14)$$

The essence of Kepler's second law is that the sector velocity has the same value at all points in the orbit. When the planet is close to the sun, r is small and $d\theta/dt$ is large; when the planet is far from the sun, r is large and $d\theta/dt$ is small.

To see how Kepler's second law follows from Newton's laws, we express dA/dt in terms of the velocity vector $\vec{v}$ of the planet P. The component of $\vec{v}$ perpendicular to the radial line is $v_\perp = v\sin\phi$. From Fig. 13.19b the displacement along the direction of $v_\perp$ during time dt is $r\,d\theta$, so we also have $v_\perp = r\,d\theta/dt$. Using this relationship in Eq. (13.14), we find

$$\frac{dA}{dt} = \frac{1}{2}rv\sin\phi \qquad \text{(sector velocity)} \qquad (13.15)$$

13.19 (a) The planet (P) moves about the sun (S) in an elliptical orbit. (b) In a time dt the line SP sweeps out an area $dA = \frac{1}{2}(r\,d\theta)r = \frac{1}{2}r^2\,d\theta$. (c) The planet's speed varies so that the line SP sweeps out the same area A in a given time t regardless of the planet's position in its orbit.



Now $rv\sin\phi$ is the magnitude of the vector product $\vec{r} \times \vec{v}$, which in turn is $1/m$ times the angular momentum $\vec{L} = \vec{r} \times m\vec{v}$ of the planet with respect to the sun. So we have

$$\frac{dA}{dt} = \frac{1}{2m}\left|\vec{r} \times m\vec{v}\right| = \frac{L}{2m} \qquad (13.16)$$

Thus Kepler's second law—that sector velocity is constant—means that angular momentum is constant!

It is easy to see why the angular momentum of the planet *must* be constant. According to Eq. (10.26), the rate of change of $\vec{L}$ equals the torque of the gravitational force $\vec{F}$ acting on the planet:

$$\frac{d\vec{L}}{dt} = \vec{\tau} = \vec{r} \times \vec{F}$$

In our situation, $\vec{r}$ is the vector from the sun to the planet, and the force $\vec{F}$ is directed from the planet to the sun. So these vectors always lie along the same line, and their vector product $\vec{r} \times \vec{F}$ is zero. Hence $d\vec{L}/dt = \mathbf{0}$. This conclusion does not depend on the $1/r^2$ behavior of the force; angular momentum is conserved for *any* force that acts always along the line joining the particle to a fixed point. Such a force is called a *central force.* (Kepler's first and third laws are valid *only* for a $1/r^2$ force.)

Conservation of angular momentum also explains why the orbit lies in a plane. The vector $\vec{L} = \vec{r} \times m\vec{v}$ is always perpendicular to the plane of the vectors $\vec{r}$ and $\vec{v}$; since $\vec{L}$ is constant in magnitude *and* direction, $\vec{r}$ and $\vec{v}$ always lie in the same plane, which is just the plane of the planet's orbit.

Application Biological Hazards of Interplanetary Travel

A spacecraft sent from earth to another planet spends most of its journey coasting along an elliptical orbit with the sun at one focus. Rockets are used only at the start and end of the journey, and even the trip to a nearby planet like Mars takes several months. During its journey, the spacecraft is exposed to cosmic rays—radiation that emanates from elsewhere in our galaxy. (On earth we're shielded from this radiation by our planet's magnetic field, as we'll describe in Chapter 27.) This poses no problem for a robotic spacecraft, but would be a severe medical hazard for astronauts undertaking such a voyage.

Kepler's Third Law

We have already derived Kepler's third law for the particular case of circular orbits. Equation (13.12) shows that the period of a satellite or planet in a circular orbit is proportional to the $\frac{3}{2}$ power of the orbit radius. Newton was able to show that this same relationship holds for an *elliptical* orbit, with the orbit radius r replaced by the semi-major axis a:

$$T = \frac{2\pi a^{3/2}}{\sqrt{Gm_S}} \quad \text{(elliptical orbit around the sun)} \qquad (13.17)$$

Since the planet orbits the sun, not the earth, we have replaced the earth's mass m_E in Eq. (13.12) with the sun's mass m_S. Note that the period does not depend on the eccentricity e. An asteroid in an elongated elliptical orbit with semi-major axis a will have the same orbital period as a planet in a circular orbit of radius a. The key difference is that the asteroid moves at different speeds at different points in its elliptical orbit (Fig. 13.19c), while the planet's speed is constant around its circular orbit.

Conceptual Example 13.7 Orbital speeds

At what point in an elliptical orbit (see Fig. 13.19) does a planet move the fastest? The slowest?

SOLUTION

Mechanical energy is conserved as a planet moves in its orbit. The planet's kinetic energy $K = \frac{1}{2}mv^2$ is maximum when the potential energy $U = -Gm_S m/r$ is minimum (that is, most negative; see Fig. 13.11), which occurs when the sun–planet distance r is a minimum. Hence the speed v is greatest at perihelion. Similarly, K is minimum when r is maximum, so the speed is slowest at aphelion.

Your intuition about falling bodies is helpful here. As the planet falls inward toward the sun, it picks up speed, and its speed is maximum when closest to the sun. The planet slows down as it moves away from the sun, and its speed is minimum at aphelion.

Example 13.8 Kepler's third law

The asteroid Pallas has an orbital period of 4.62 years and an orbital eccentricity of 0.233. Find the semi-major axis of its orbit.

SOLUTION

IDENTIFY and SET UP: This example uses Kepler's third law, which relates the period T and the semi-major axis a for an orbiting object (such as an asteroid). We use Eq. (13.17) to determine a; from Appendix F we have $m_S = 1.99 \times 10^{30}$ kg, and a conversion factor from Appendix E gives $T = (4.62\text{ yr})(3.156 \times 10^7\text{ s/yr}) = 1.46 \times 10^8$ s. Note that we don't need the value of the eccentricity.

EXECUTE: From Eq. (13.17), $a^{3/2} = [(Gm_S)^{1/2}T]/2\pi$. To solve for a, we raise both sides of this expression to the $\frac{2}{3}$ power and then substitute the values of G, m_S, and T:

$$a = \left(\frac{Gm_S T^2}{4\pi^2}\right)^{1/3} = 4.15 \times 10^{11}\text{ m}$$

(Plug in the numbers yourself to check.)

EVALUATE: Our result is intermediate between the semi-major axes of Mars and Jupiter (see Appendix F). Most known asteroids orbit in an "asteroid belt" between the orbits of these two planets.

Example 13.9 Comet Halley

Comet Halley moves in an elongated elliptical orbit around the sun (Fig. 13.20). Its distances from the sun at perihelion and aphelion are 8.75×10^7 km and 5.26×10^9 km, respectively. Find the orbital semi-major axis, eccentricity, and period.

SOLUTION

IDENTIFY and SET UP: We are to find the semi-major axis a, eccentricity e, and orbital period T. We can use Fig. 13.18 to find a and e from the given perihelion and aphelion distances. Knowing a, we can find T from Kepler's third law, Eq. (13.17).

EXECUTE: From Fig. 13.18, the length $2a$ of the major axis equals the sum of the comet–sun distance at perihelion and the comet–sun distance at aphelion. Hence

$$a = \frac{(8.75 \times 10^7\text{ km}) + (5.26 \times 10^9\text{ km})}{2} = 2.67 \times 10^9\text{ km}$$

Figure 13.19 also shows that the comet–sun distance at perihelion is $a - ea = a(1 - e)$. This distance is 8.75×10^7 km, so

$$e = 1 - \frac{8.75 \times 10^7\text{ km}}{a} = 1 - \frac{8.75 \times 10^7\text{ km}}{2.67 \times 10^9\text{ km}} = 0.967$$

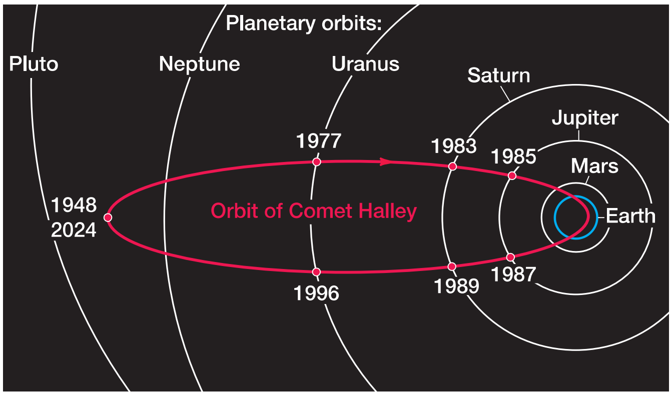
From Eq. (13.17), the period is

$$T = \frac{2\pi a^{3/2}}{\sqrt{Gm_S}} = \frac{2\pi(2.67 \times 10^{12}\text{ m})^{3/2}}{\sqrt{(6.67 \times 10^{-11}\text{ N}\cdot\text{m}^2/\text{kg}^2)(1.99 \times 10^{30}\text{ kg})}}$$
$$= 2.38 \times 10^9\text{ s} = 75.5\text{ years}$$

EVALUATE: The eccentricity is close to 1, so the orbit is very elongated (see Fig. 13.20a). Comet Halley was at perihelion in early 1986 (Fig. 13.20b); it will next reach perihelion one period later, in 2061.

13.20 (a) The orbit of Comet Halley. (b) Comet Halley as it appeared in 1986. At the heart of the comet is an icy body, called the nucleus, that is about 10 km across. When the comet's orbit carries it close to the sun, the heat of sunlight causes the nucleus to partially evaporate. The evaporated material forms the tail, which can be tens of millions of kilometers long.

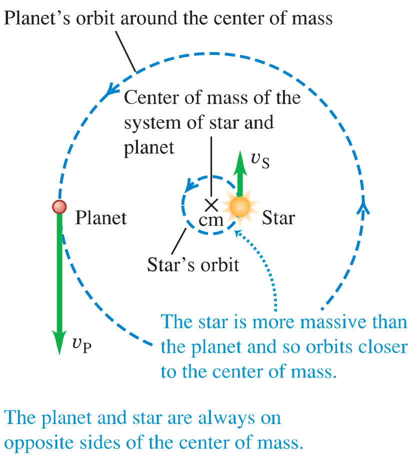
(a)



(b)

Planetary Motions and the Center of Mass

We have assumed that as a planet or comet orbits the sun, the sun remains absolutely stationary. Of course, this can't be correct; because the sun exerts a



13.21 A star and its planet both orbit about their common center of mass.

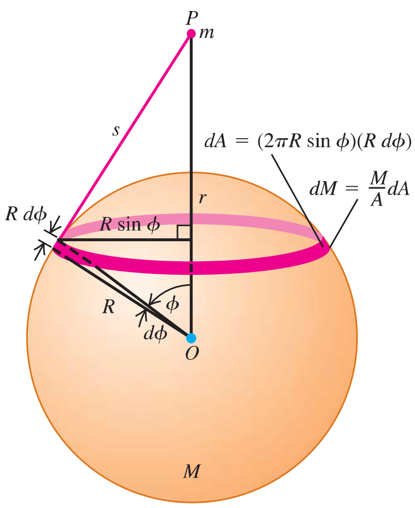
gravitational force on the planet, the planet exerts a gravitational force on the sun of the same magnitude but opposite direction. In fact, *both* the sun and the planet orbit around their common center of mass (Fig. 13.21). We've made only a small error by ignoring this effect, however; the sun's mass is about 750 times the total mass of all the planets combined, so the center of mass of the solar system is not far from the center of the sun. Remarkably, astronomers have used this effect to detect the presence of planets orbiting other stars. Sensitive telescopes are able to detect the apparent "wobble" of a star as it orbits the common center of mass of the star and an unseen companion planet. (The planets are too faint to observe directly.) By analyzing these "wobbles," astronomers have discovered planets in orbit around hundreds of other stars.

Newton's analysis of planetary motions is used on a daily basis by modern-day astronomers. But the most remarkable result of Newton's work is that the motions of bodies in the heavens obey the *same* laws of motion as do bodies on the earth. This *Newtonian synthesis,* as it has come to be called, is one of the great unifying principles of science. It has had profound effects on the way that humanity looks at the universe—not as a realm of impenetrable mystery, but as a direct extension of our everyday world, subject to scientific study and calculation.

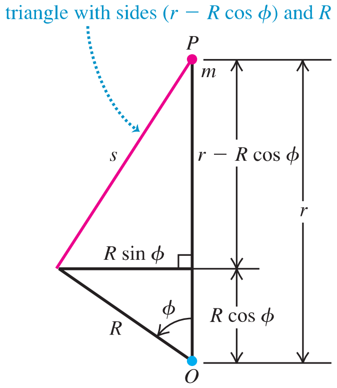
Test Your Understanding of Section 13.5 The orbit of Comet X has a semi-major axis that is four times longer than the semi-major axis of Comet Y. What is the ratio of the orbital period of X to the orbital period of Y? (i) 2; (ii) 4; (iii) 8; (iv) 16; (v) 32; (vi) 64.

13.22 Calculating the gravitational potential energy of interaction between a point mass m outside a spherical shell and a ring on the surface of the shell.

(a) Geometry of the situation



(b) The distance s is the hypotenuse of a right triangle with sides $(r - R\cos\phi)$ and $R\sin\phi$.



13.6 Spherical Mass Distributions

We have stated without proof that the gravitational interaction between two spherically symmetric mass distributions is the same as though all the mass of each were concentrated at its center. Now we're ready to prove this statement. Newton searched for a proof for several years, and he delayed publication of the law of gravitation until he found one.

Here's our program. Rather than starting with two spherically symmetric masses, we'll tackle the simpler problem of a point mass m interacting with a thin spherical shell with total mass M. We will show that when m is outside the sphere, the *potential energy* associated with this gravitational interaction is the same as though M were all concentrated at the center of the sphere. We learned in Section 7.4 that the force is the negative derivative of the potential energy, so the *force* on m is also the same as for a point mass M. Any spherically symmetric mass distribution can be thought of as being made up of many concentric spherical shells, so our result will also hold for *any* spherically symmetric M.

A Point Mass Outside a Spherical Shell

We start by considering a ring on the surface of the shell (Fig. 13.22a), centered on the line from the center of the shell to m. We do this because all of the particles that make up the ring are the same distance s from the point mass m. From Eq. (13.9) the potential energy of interaction between the earth (mass m_E) and a point mass m, separated by a distance r, is $U = -Gm_E m/r$. By changing notation in this expression, we see that the potential energy of interaction between the point mass m and a particle of mass m_i within the ring is given by

$$U_i = -\frac{Gmm_i}{s}$$

To find the potential energy of interaction between m and the entire ring of mass $dM = \sum_i m_i$, we sum this expression for U_i over all particles in the ring. Calling this potential energy dU, we find

$$dU = \sum_i U_i = \sum_i \left(-\frac{Gmm_i}{s}\right) = -\frac{Gm}{s}\sum_i m_i = -\frac{Gm\,dM}{s} \quad (13.18)$$

To proceed, we need to know the mass dM of the ring. We can find this with the aid of a little geometry. The radius of the shell is R, so in terms of the angle ϕ shown in the figure, the radius of the ring is $R\sin\phi$, and its circumference is $2\pi R\sin\phi$. The width of the ring is $R\,d\phi$, and its area dA is approximately equal to its width times its circumference:

$$dA = 2\pi R^2 \sin\phi\, d\phi$$

The ratio of the ring mass dM to the total mass M of the shell is equal to the ratio of the area dA of the ring to the total area $A = 4\pi R^2$ of the shell:

$$\frac{dM}{M} = \frac{2\pi R^2 \sin\phi\, d\phi}{4\pi R^2} = \tfrac{1}{2}\sin\phi\, d\phi \quad (13.19)$$

Now we solve Eq. (13.19) for dM and substitute the result into Eq. (13.18) to find the potential energy of interaction between the point mass m and the ring:

$$dU = -\frac{GMm\sin\phi\, d\phi}{2s} \quad (13.20)$$

The total potential energy of interaction between the point mass and the *shell* is the integral of Eq. (13.20) over the whole sphere as ϕ varies from 0 to π (*not* 2π!) and s varies from $r - R$ to $r + R$. To carry out the integration, we have to express the integrand in terms of a single variable; we choose s. To express ϕ and $d\phi$ in terms of s, we have to do a little more geometry. Figure 13.22b shows that s is the hypotenuse of a right triangle with sides $(r - R\cos\phi)$ and $R\sin\phi$, so the Pythagorean theorem gives

$$\begin{aligned} s^2 &= (r - R\cos\phi)^2 + (R\sin\phi)^2 \\ &= r^2 - 2rR\cos\phi + R^2 \end{aligned} \quad (13.21)$$

We take differentials of both sides:

$$2s\,ds = 2rR\sin\phi\, d\phi$$

Next we divide this by $2rR$ and substitute the result into Eq. (13.20):

$$dU = -\frac{GMm}{2s}\frac{s\,ds}{rR} = -\frac{GMm}{2rR}\,ds \quad (13.22)$$

We can now integrate Eq. (13.22), recalling that s varies from $r - R$ to $r + R$:

$$U = -\frac{GMm}{2rR}\int_{r-R}^{r+R} ds = -\frac{GMm}{2rR}\big[(r + R) - (r - R)\big] \quad (13.23)$$

Finally, we have

$$U = -\frac{GMm}{r} \quad \text{(point mass } m \text{ outside spherical shell } M) \quad (13.24)$$

This is equal to the potential energy of two point masses m and M at a distance r. So we have proved that the gravitational potential energy of the spherical shell M and the point mass m at any distance r is the same as though they were point masses. Because the force is given by $F_r = -dU/dr$, the force is also the same.

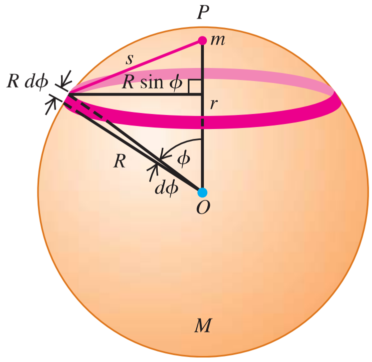
The Gravitational Force Between Spherical Mass Distributions

Any spherically symmetric mass distribution can be thought of as a combination of concentric spherical shells. Because of the principle of superposition of forces, what is true of one shell is also true of the combination. So we have proved half of what we set out to prove: that the gravitational interaction between any spherically symmetric mass distribution and a point mass is the same as though all the mass of the spherically symmetric distribution were concentrated at its center.

The other half is to prove that *two* spherically symmetric mass distributions interact as though they were both points. That's easier. In Fig. 13.22a the forces the two bodies exert on each other are an action–reaction pair, and they obey Newton's third law. So we have also proved that the force that m exerts *on* the sphere M is the same as though M were a point. But now if we replace m with a spherically symmetric mass distribution centered at m's location, the resulting gravitational force on any part of M is the same as before, and so is the total force. This completes our proof.

A Point Mass Inside a Spherical Shell

13.23 When a point mass m is *inside* a uniform spherical shell of mass M, the potential energy is the same no matter where inside the shell the point mass is located. The force from the masses' mutual gravitational interaction is zero.



We assumed at the beginning that the point mass m was outside the spherical shell, so our proof is valid only when m is outside a spherically symmetric mass distribution. When m is *inside* a spherical shell, the geometry is as shown in Fig. 13.23. The entire analysis goes just as before; Eqs. (13.18) through (13.22) are still valid. But when we get to Eq. (13.23), the limits of integration have to be changed to $R - r$ and $R + r$. We then have

$$U = -\frac{GMm}{2rR}\int_{R-r}^{R+r} ds = -\frac{GMm}{2rR}[(R + r) - (R - r)] \qquad (13.25)$$

and the final result is

$$U = -\frac{GMm}{R} \qquad \text{(point mass } m \text{ inside spherical shell } M\text{)} \qquad (13.26)$$

Compare this result to Eq. (13.24): Instead of having r, the distance between m and the center of M, in the denominator, we have R, the radius of the shell. This means that U in Eq. (13.26) doesn't depend on r and thus has the same value everywhere inside the shell. When m moves around inside the shell, no work is done on it, so the force on m at any point inside the shell must be zero.

More generally, at any point in the interior of any spherically symmetric mass distribution (not necessarily a shell), at a distance r from its center, the gravitational force on a point mass m is the same as though we removed all the mass at points farther than r from the center and concentrated all the remaining mass at the center.

Example 13.10 "Journey to the center of the earth"

Imagine that we drill a hole through the earth along a diameter and drop a mail pouch down the hole. Derive an expression for the gravitational force F_g on the pouch as a function of its distance from the earth's center. Assume that the earth's density is uniform (not a very realistic model; see Fig. 13.9).

SOLUTION

IDENTIFY and SET UP: From the discussion immediately above, the value of F_g at a distance r from the earth's center is determined only by the mass M within a spherical region of radius r (Fig. 13.24). Hence F_g is the same as if all the mass within radius r were concentrated at the center of the earth. The mass of a uniform sphere is proportional to the volume of the sphere, which is $\frac{4}{3}\pi r^3$ for a sphere of arbitrary radius r and $\frac{4}{3}\pi R_E^3$ for the entire earth.

EXECUTE: The ratio of the mass M of the sphere of radius r to the mass m_E of the earth is

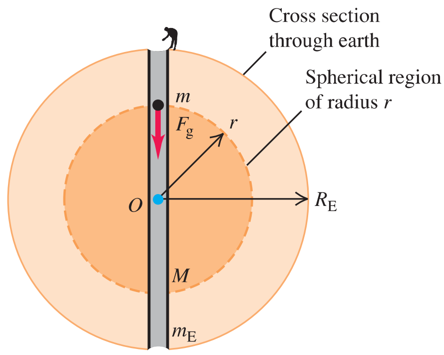
$$\frac{M}{m_E} = \frac{\frac{4}{3}\pi r^3}{\frac{4}{3}\pi R_E^3} = \frac{r^3}{R_E^3} \qquad \text{so} \qquad M = m_E\frac{r^3}{R_E^3}$$

The magnitude of the gravitational force on m is then

$$F_g = \frac{GMm}{r^2} = \frac{Gm}{r^2}\left(m_E \frac{r^3}{R_E^3}\right) = \frac{Gm_E m}{R_E^3} r$$

EVALUATE: Inside this uniform-density sphere, F_g is *directly proportional* to the distance r from the center, rather than to $1/r^2$ as it is outside the sphere. At the surface $r = R_E$, we have $F_g = Gm_E m/R_E^2$, as we should. In the next chapter we'll learn how to compute the time it would take for the mail pouch to emerge on the other side of the earth.

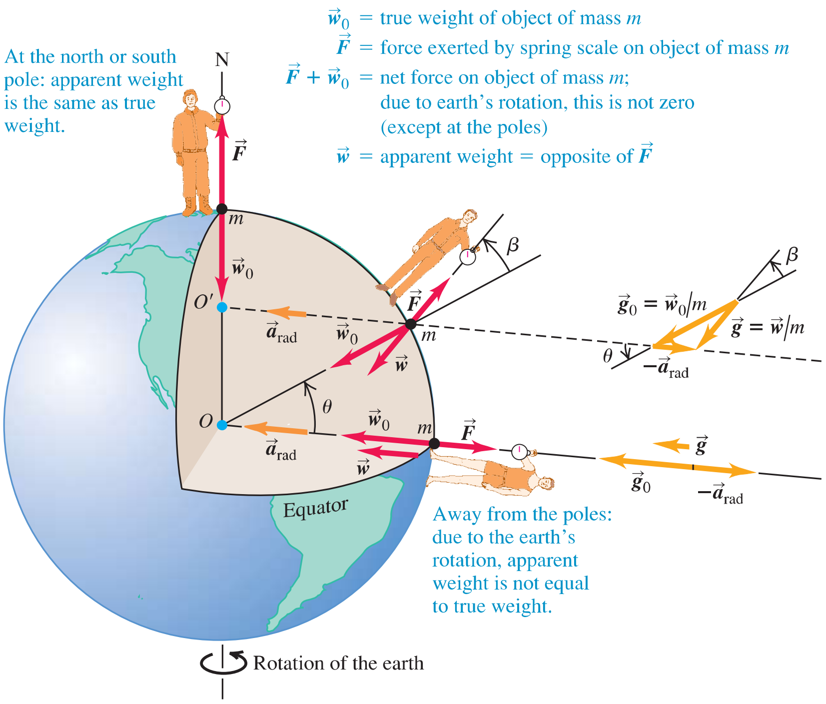
13.24 A hole through the center of the earth (assumed to be uniform). When an object is a distance r from the center, only the mass inside a sphere of radius r exerts a net gravitational force on it.



Test Your Understanding of Section 13.6 In the classic 1913 science-fiction novel *At the Earth's Core* by Edgar Rice Burroughs, explorers discover that the earth is a hollow sphere and that an entire civilization lives on the inside of the sphere. Would it be possible to stand and walk on the inner surface of a hollow, nonrotating planet?

13.7 Apparent Weight and the Earth's Rotation

Because the earth rotates on its axis, it is not precisely an inertial frame of reference. For this reason the apparent weight of a body on earth is not precisely equal to the earth's gravitational attraction, which we will call the **true weight** $\vec{w}_0$ of the body. Figure 13.25 is a cutaway view of the earth, showing three observers. Each one holds a spring scale with a body of mass m hanging from it. Each scale applies a tension force $\vec{F}$ to the body hanging from it, and the reading on each scale is the magnitude F of this force. If the observers are unaware of the earth's



13.25 Except at the poles, the reading for an object being weighed on a scale (the *apparent weight*) is less than the gravitational force of attraction on the object (the *true weight*). The reason is that a net force is needed to provide a centripetal acceleration as the object rotates with the earth. For clarity, the illustration greatly exaggerates the angle β between the true and apparent weight vectors.

rotation, each one *thinks* that the scale reading equals the weight of the body because he thinks the body on his spring scale is in equilibrium. So each observer thinks that the tension $\vec{F}$ must be opposed by an equal and opposite force $\vec{w}$, which we call the **apparent weight.** But if the bodies are rotating with the earth, they are *not* precisely in equilibrium. Our problem is to find the relationship between the apparent weight $\vec{w}$ and the true weight $\vec{w}_0$.

If we assume that the earth is spherically symmetric, then the true weight $\vec{w}_0$ has magnitude Gm_Em/R_E^2, where m_E and R_E are the mass and radius of the earth. This value is the same for all points on the earth's surface. If the center of the earth can be taken as the origin of an inertial coordinate system, then the body at the north pole really *is* in equilibrium in an inertial system, and the reading on that observer's spring scale is equal to w_0. But the body at the equator is moving in a circle of radius R_E with speed v, and there must be a net inward force equal to the mass times the centripetal acceleration:

$$w_0 - F = \frac{mv^2}{R_E}$$

So the magnitude of the apparent weight (equal to the magnitude of F) is

$$w = w_0 - \frac{mv^2}{R_E} \qquad \text{(at the equator)} \tag{13.27}$$

If the earth were not rotating, the body when released would have a free-fall acceleration $g_0 = w_0/m$. Since the earth *is* rotating, the falling body's actual acceleration relative to the observer at the equator is $g = w/m$. Dividing Eq. (13.27) by m and using these relationships, we find

$$g = g_0 - \frac{v^2}{R_E} \qquad \text{(at the equator)}$$

To evaluate v^2/R_E, we note that in 86,164 s a point on the equator moves a distance equal to the earth's circumference, $2\pi R_E = 2\pi(6.38 \times 10^6\ \text{m})$. (The solar day, 86,400 s, is $\frac{1}{365}$ longer than this because in one day the earth also completes $\frac{1}{365}$ of its orbit around the sun.) Thus we find

$$v = \frac{2\pi(6.38 \times 10^6\ \text{m})}{86{,}164\ \text{s}} = 465\ \text{m/s}$$

$$\frac{v^2}{R_E} = \frac{(465\ \text{m/s})^2}{6.38 \times 10^6\ \text{m}} = 0.0339\ \text{m/s}^2$$

So for a spherically symmetric earth the acceleration due to gravity should be about $0.03\ \text{m/s}^2$ less at the equator than at the poles.

At locations intermediate between the equator and the poles, the true weight $\vec{w}_0$ and the centripetal acceleration are not along the same line, and we need to write a vector equation corresponding to Eq. (13.27). From Fig. 13.25 we see that the appropriate equation is

$$\vec{w} = \vec{w}_0 - m\vec{a}_{\text{rad}} = m\vec{g}_0 - m\vec{a}_{\text{rad}} \tag{13.28}$$

The difference in the magnitudes of g and g_0 lies between zero and $0.0339\ \text{m/s}^2$. As shown in Fig. 13.25, the *direction* of the apparent weight differs from the direction toward the center of the earth by a small angle β, which is 0.1° or less.

Table 13.1 gives the values of g at several locations, showing variations with latitude. There are also small additional variations due to the lack of perfect spherical symmetry of the earth, local variations in density, and differences in elevation.

Table 13.1 Variations of g with Latitude and Elevation

Station	North Latitude	Elevation (m)	$g(\text{m/s}^2)$
Canal Zone	09°	0	9.78243
Jamaica	18°	0	9.78591
Bermuda	32°	0	9.79806
Denver, CO	40°	1638	9.79609
Pittsburgh, PA	40.5°	235	9.80118
Cambridge, MA	42°	0	9.80398
Greenland	70°	0	9.82534

Test Your Understanding of Section 13.7 Imagine a planet that has the same mass and radius as the earth, but that makes 10 rotations during the time the earth makes one rotation. What would be the difference between the acceleration due to gravity at the planet's equator and the acceleration due to gravity at its poles? (i) $0.00339\ \mathrm{m/s^2}$; (ii) $0.0339\ \mathrm{m/s^2}$; (iii) $0.339\ \mathrm{m/s^2}$; (iv) $3.39\ \mathrm{m/s^2}$.

13.8 Black Holes

The concept of a black hole is one of the most interesting and startling products of modern gravitational theory, yet the basic idea can be understood on the basis of Newtonian principles.

The Escape Speed from a Star

Think first about the properties of our own sun. Its mass $M = 1.99 \times 10^{30}$ kg and radius $R = 6.96 \times 10^8$ m are much larger than those of any planet, but compared to other stars, our sun is not exceptionally massive. You can find the sun's average density ρ in the same way we found the average density of the earth in Section 13.2:

$$\rho = \frac{M}{V} = \frac{M}{\frac{4}{3}\pi R^3} = \frac{1.99 \times 10^{30}\ \mathrm{kg}}{\frac{4}{3}\pi(6.96 \times 10^8\ \mathrm{m})^3} = 1410\ \mathrm{kg/m^3}$$

The sun's temperatures range from 5800 K (about 5500°C or 10,000°F) at the surface up to 1.5×10^7 K (about 2.7×10^7°F) in the interior, so it surely contains no solids or liquids. Yet gravitational attraction pulls the sun's gas atoms together until the sun is, on average, 41% denser than water and about 1200 times as dense as the air we breathe.

Now think about the escape speed for a body at the surface of the sun. In Example 13.5 (Section 13.3) we found that the escape speed from the surface of a spherical mass M with radius R is $v = \sqrt{2GM/R}$. We can relate this to the average density. Substituting $M = \rho V = \rho(\frac{4}{3}\pi R^3)$ into the expression for escape speed gives

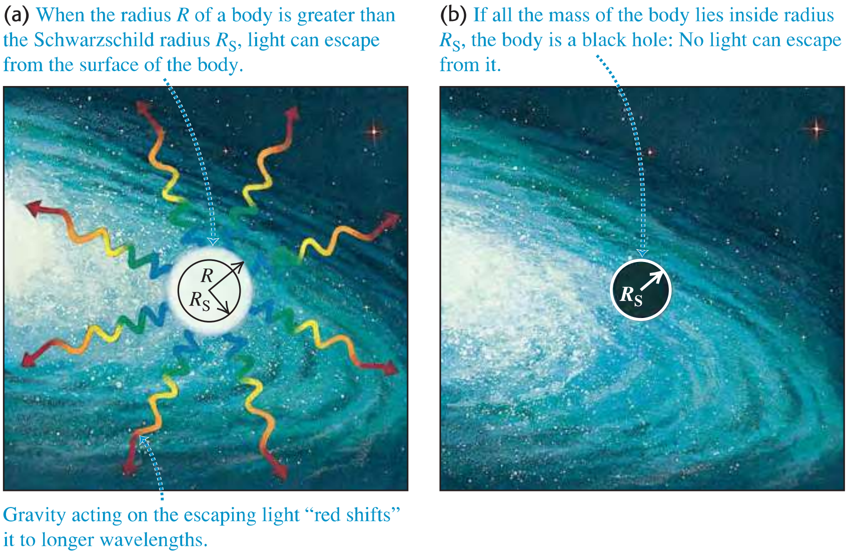
$$v = \sqrt{\frac{2GM}{R}} = \sqrt{\frac{8\pi G\rho}{3}}R \tag{13.29}$$

Using either form of this equation, you can show that the escape speed for a body at the surface of our sun is $v = 6.18 \times 10^5$ m/s (about 2.2 million km/h, or 1.4 million mi/h). This value, roughly $\frac{1}{500}$ the speed of light, is independent of the mass of the escaping body; it depends on only the mass and radius (or average density and radius) of the sun.

Now consider various stars with the same average density ρ and different radii R. Equation (13.29) shows that for a given value of density ρ, the escape speed v is directly proportional to R. In 1783 the Rev. John Mitchell, an amateur astronomer, noted that if a body with the same average density as the sun had about 500 times the radius of the sun, its escape speed would be greater than the speed of light c. With his statement that "all light emitted from such a body would be made to return toward it," Mitchell became the first person to suggest the existence of what we now call a **black hole**—an object that exerts a gravitational force on other bodies but cannot emit any light of its own.

Black Holes, the Schwarzschild Radius, and the Event Horizon

The first expression for escape speed in Eq. (13.29) suggests that a body of mass M will act as a black hole if its radius R is less than or equal to a certain critical radius. How can we determine this critical radius? You might think that you can find the answer by simply setting $v = c$ in Eq. (13.29). As a matter of fact, this does give the correct result, but only because of two compensating errors.



13.26 (a) A body with a radius R greater than the Schwarzschild radius R_S. (b) If the body collapses to a radius smaller than R_S, it is a black hole with an escape speed greater than the speed of light. The surface of the sphere of radius R_S is called the event horizon of the black hole.

The kinetic energy of light is *not* $mc^2/2$, and the gravitational potential energy near a black hole is *not* given by Eq. (13.9). In 1916, Karl Schwarzschild used Einstein's general theory of relativity (in part a generalization and extension of Newtonian gravitation theory) to derive an expression for the critical radius R_S, now called the **Schwarzschild radius.** The result turns out to be the same as though we had set $v = c$ in Eq. (13.29), so

$$c = \sqrt{\frac{2GM}{R_S}}$$

Solving for the Schwarzschild radius R_S, we find

$$R_S = \frac{2GM}{c^2} \qquad \text{(Schwarzschild radius)} \qquad (13.30)$$

If a spherical, nonrotating body with mass M has a radius less than R_S, then *nothing* (not even light) can escape from the surface of the body, and the body is a black hole (Fig. 13.26). In this case, any other body within a distance R_S of the center of the black hole is trapped by the gravitational attraction of the black hole and cannot escape from it.

The surface of the sphere with radius R_S surrounding a black hole is called the **event horizon:** Since light can't escape from within that sphere, we can't see events occurring inside. All that an observer outside the event horizon can know about a black hole is its mass (from its gravitational effects on other bodies), its electric charge (from the electric forces it exerts on other charged bodies), and its angular momentum (because a rotating black hole tends to drag space—and everything in that space—around with it). All other information about the body is irretrievably lost when it collapses inside its event horizon.

Example 13.11 Black hole calculations

Astrophysical theory suggests that a burned-out star whose mass is at least three solar masses will collapse under its own gravity to form a black hole. If it does, what is the radius of its event horizon?

SOLUTION

IDENTIFY, SET UP, and EXECUTE: The radius in question is the Schwarzschild radius. We use Eq. (13.30) with a value of M equal to three solar masses, or $M = 3(1.99 \times 10^{30}\ \text{kg}) = 6.0 \times 10^{30}$ kg:

$$R_S = \frac{2GM}{c^2} = \frac{2(6.67 \times 10^{-11}\ \text{N} \cdot \text{m}^2/\text{kg}^2)(6.0 \times 10^{30}\ \text{kg})}{(3.00 \times 10^8\ \text{m/s})^2}$$

$$= 8.9 \times 10^3\ \text{m} = 8.9\ \text{km} = 5.5\ \text{mi}$$

EVALUATE: The average density of such an object is

$$\rho = \frac{M}{\frac{4}{3}\pi R^3} = \frac{6.0 \times 10^{30}\ \text{kg}}{\frac{4}{3}\pi(8.9 \times 10^3\ \text{m})^3} = 2.0 \times 10^{18}\ \text{kg/m}^3$$

This is about 10^{15} times as great as the density of familiar matter on earth and is comparable to the densities of atomic nuclei. In fact, once the body collapses to a radius of R_S, nothing can prevent it from collapsing further. All of the mass ends up being crushed down to a single point called a *singularity* at the center of the event horizon. This point has zero volume and so has *infinite* density.

A Visit to a Black Hole

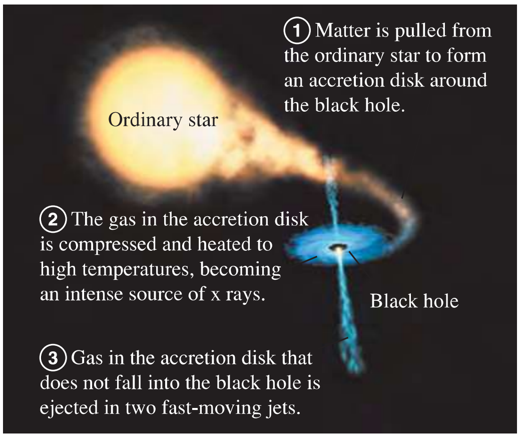
At points far from a black hole, its gravitational effects are the same as those of any normal body with the same mass. If the sun collapsed to form a black hole, the orbits of the planets would be unaffected. But things get dramatically different close to the black hole. If you decided to become a martyr for science and jump into a black hole, the friends you left behind would notice several odd effects as you moved toward the event horizon, most of them associated with effects of general relativity.

If you carried a radio transmitter to send back your comments on what was happening, your friends would have to retune their receiver continuously to lower and lower frequencies, an effect called the *gravitational red shift*. Consistent with this shift, they would observe that your clocks (electronic or biological) would appear to run more and more slowly, an effect called *time dilation*. In fact, during their lifetimes they would never see you make it to the event horizon.

In your frame of reference, you would make it to the event horizon in a rather short time but in a rather disquieting way. As you fell feet first into the black hole, the gravitational pull on your feet would be greater than that on your head, which would be slightly farther away from the black hole. The *differences* in gravitational force on different parts of your body would be great enough to stretch you along the direction toward the black hole and compress you perpendicular to it. These effects (called *tidal forces*) would rip you to atoms, and then rip your atoms apart, before you reached the event horizon.

Detecting Black Holes

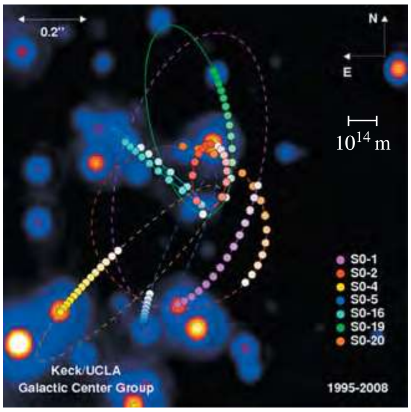
If light cannot escape from a black hole and if black holes are as small as Example 13.11 suggests, how can we know that such things exist? The answer is that any gas or dust near the black hole tends to be pulled into an *accretion disk* that swirls around and into the black hole, rather like a whirlpool (Fig. 13.27). Friction within the accretion disk's material causes it to lose mechanical energy



13.27 A binary star system in which an ordinary star and a black hole orbit each other. The black hole itself cannot be seen, but the x rays from its accretion disk can be detected.

and spiral into the black hole; as it moves inward, it is compressed together. This causes heating of the material, just as air compressed in a bicycle pump gets hotter. Temperatures in excess of 10^6 K can occur in the accretion disk, so hot that the disk emits not just visible light (as do bodies that are "red-hot" or "white-hot") but x rays. Astronomers look for these x rays (emitted by the material *before* it crosses the event horizon) to signal the presence of a black hole. Several promising candidates have been found, and astronomers now express considerable confidence in the existence of black holes.

13.28 This false-color image shows the motions of stars at the center of our galaxy over a 13-year period. Analyzing these orbits using Kepler's third law indicates that the stars are moving about an unseen object that is some 4.1×10^6 times the mass of the sun. The scale bar indicates a length of 10^{14} m (670 times the distance from the earth to the sun) at the distance of the galactic center.



Black holes in binary star systems like the one depicted in Fig. 13.27 have masses a few times greater than the sun's mass. There is also mounting evidence for the existence of much larger *supermassive black holes.* One example is thought to lie at the center of our Milky Way galaxy, some 26,000 light-years from earth in the direction of the constellation Sagittarius. High-resolution images of the galactic center reveal stars moving at speeds greater than 1500 km/s about an unseen object that lies at the position of a source of radio waves called Sgr A* (Fig. 13.28). By analyzing these motions, astronomers can infer the period T and semi-major axis a of each star's orbit. The mass m_X of the unseen object can then be calculated using Kepler's third law in the form given in Eq. (13.17), with the mass of the sun m_S replaced by m_X:

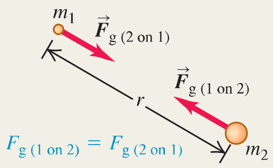
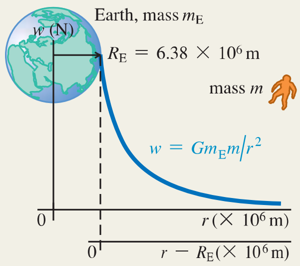
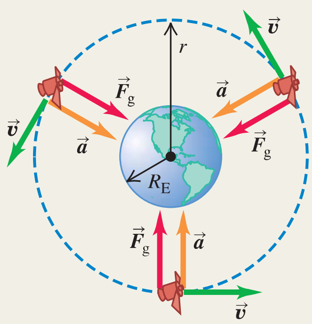
$$T = \frac{2\pi a^{3/2}}{\sqrt{Gm_X}} \qquad \text{so} \qquad m_X = \frac{4\pi^2 a^3}{GT^2}$$

The conclusion is that the mysterious dark object at the galactic center has a mass of 8.2×10^{36} kg, or 4.1 *million* times the mass of the sun. Yet observations with radio telescopes show that it has a radius no more than 4.4×10^{10} m, about one-third of the distance from the earth to the sun. These observations suggest that this massive, compact object is a black hole with a Schwarzschild radius of 1.1×10^{10} m. Astronomers hope to improve the resolution of their observations so that they can actually see the event horizon of this black hole.

Other lines of research suggest that even larger black holes, in excess of 10^9 times the mass of the sun, lie at the centers of other galaxies. Observational and theoretical studies of black holes of all sizes continue to be an exciting area of research in both physics and astronomy.

Test Your Understanding of Section 13.8 If the sun somehow collapsed to form a black hole, what effect would this event have on the orbit of the earth? (i) The orbit would shrink; (ii) the orbit would expand; (iii) the orbit would remain the same size.

CHAPTER 13 SUMMARY

Newton's law of gravitation: *Any* two bodies with masses m_1 and m_2, a distance r apart, attract each other with forces inversely proportional to r^2. These forces form an action–reaction pair and obey Newton's third law. When two or more bodies exert gravitational forces on a particular body, the total gravitational force on that individual body is the vector sum of the forces exerted by the other bodies. The gravitational interaction between spherical mass distributions, such as planets or stars, is the same as if all the mass of each distribution were concentrated at the center. (See Examples 13.1–13.3 and 13.10.)	$F_g = \dfrac{Gm_1m_2}{r^2}$	(13.1)	
Gravitational force, weight, and gravitational potential energy: The weight w of a body is the total gravitational force exerted on it by all other bodies in the universe. Near the surface of the earth (mass m_E and radius R_E), the weight is essentially equal to the gravitational force of the earth alone. The gravitational potential energy U of two masses m and m_E separated by a distance r is inversely proportional to r. The potential energy is never positive; it is zero only when the two bodies are infinitely far apart. (See Examples 13.4 and 13.5.)	$w = F_g = \dfrac{Gm_Em}{R_E^2}$ (weight at earth's surface) $g = \dfrac{Gm_E}{R_E^2}$ (acceleration due to gravity at earth's surface) $U = -\dfrac{Gm_Em}{r}$	(13.3) (13.4) (13.9)	
Orbits: When a satellite moves in a circular orbit, the centripetal acceleration is provided by the gravitational attraction of the earth. Kepler's three laws describe the more general case: an elliptical orbit of a planet around the sun or a satellite around a planet. (See Examples 13.6–13.9.)	$v = \sqrt{\dfrac{Gm_E}{r}}$ (speed in circular orbit) $T = \dfrac{2\pi r}{v} = 2\pi r\sqrt{\dfrac{r}{Gm_E}} = \dfrac{2\pi r^{3/2}}{\sqrt{Gm_E}}$ (period in circular orbit)	(13.10) (13.12)	
Black holes: If a nonrotating spherical mass distribution with total mass M has a radius less than its Schwarzschild radius R_S, it is called a black hole. The gravitational interaction prevents anything, including light, from escaping from within a sphere with radius R_S. (See Example 13.11.)	$R_S = \dfrac{2GM}{c^2}$ (Schwarzschild radius)	(13.30)	 If all of the body is inside its Schwarzschild radius $R_S = 2GM/c^2$, the body is a black hole.

BRIDGING PROBLEM Speeds in an Elliptical Orbit

A comet orbits the sun (mass m_S) in an elliptical orbit of semi-major axis a and eccentricity e. (a) Find expressions for the speeds of the comet at perihelion and aphelion. (b) Evaluate these expressions for Comet Halley (see Example 13.9).

SOLUTION GUIDE

See MasteringPhysics® study area for a Video Tutor solution.

IDENTIFY and SET UP

1. Sketch the situation; show all relevant dimensions. Label the perihelion and aphelion.
2. List the unknown quantities, and identify the target variables.
3. Just as for a satellite orbiting the earth, the mechanical energy is conserved for a comet orbiting the sun. (Why?) What other quantity is conserved as the comet moves in its orbit? (*Hint:* See Section 13.5.)

EXECUTE

4. You'll need at least two equations that involve the two unknown speeds, and you'll need expressions for the sun–comet distances at perihelion and aphelion. (*Hint:* See Fig. 13.18.)
5. Solve the equations for your target variables. Compare your expressions: Which speed is lower? Does this make sense?
6. Use your expressions from step 5 to find the perihelion and aphelion speeds for Comet Halley. (*Hint:* See Appendix F.)

EVALUATE

7. Check whether your results make sense for the special case of a circular orbit ($e = 0$).

Problems

For instructor-assigned homework, go to www.masteringphysics.com

•, ••, •••: Problems of increasing difficulty. **CP**: Cumulative problems incorporating material from earlier chapters. **CALC**: Problems requiring calculus. **BIO**: Biosciences problems.

DISCUSSION QUESTIONS

Q13.1 A student wrote: "The only reason an apple falls downward to meet the earth instead of the earth rising upward to meet the apple is that the earth is much more massive and so exerts a much greater pull." Please comment.

Q13.2 A planet makes a circular orbit with period T around a star. If it were to orbit, at the same distance, a star with three times the mass of the original star, would the new period (in terms of T) be (a) $3T$, (b) $T\sqrt{3}$, (c) T, (d) $T/\sqrt{3}$, or (e) $T/3$?

Q13.3 If all planets had the same average density, how would the acceleration due to gravity at the surface of a planet depend on its radius?

Q13.4 Is a pound of butter on the earth the same amount as a pound of butter on Mars? What about a kilogram of butter? Explain.

Q13.5 Example 13.2 (Section 13.1) shows that the acceleration of each sphere caused by the gravitational force is inversely proportional to the mass of that sphere. So why does the force of gravity give all masses the same acceleration when they are dropped near the surface of the earth?

Q13.6 When will you attract the sun more: today at noon, or tonight at midnight? Explain.

Q13.7 Since the moon is constantly attracted toward the earth by the gravitational interaction, why doesn't it crash into the earth?

Q13.8 A planet makes a circular orbit with period T around a star. If the planet were to orbit at the same distance around this star, but had three times as much mass, what would the new period (in terms of T) be: (a) $3T$, (b) $T\sqrt{3}$, (c) T, (d) $T/\sqrt{3}$, or (e) $T/3$?

Q13.9 The sun pulls on the moon with a force that is more than twice the magnitude of the force with which the earth attracts the moon. Why, then, doesn't the sun take the moon away from the earth?

Q13.10 As defined in Chapter 7, gravitational potential energy is $U = mgy$ and is positive for a body of mass m above the earth's surface (which is at $y = 0$). But in this chapter, gravitational potential energy is $U = -Gm_E m/r$, which is *negative* for a body of mass m above the earth's surface (which is at $r = R_E$). How can you reconcile these seemingly incompatible descriptions of gravitational potential energy?

Q13.11 A planet is moving at constant speed in a circular orbit around a star. In one complete orbit, what is the net amount of work done on the planet by the star's gravitational force: positive, negative, or zero? What if the planet's orbit is an ellipse, so that the speed is not constant? Explain your answers.

Q13.12 Does the escape speed for an object at the earth's surface depend on the direction in which it is launched? Explain. Does your answer depend on whether or not you include the effects of air resistance?

Q13.13 If a projectile is fired straight up from the earth's surface, what would happen if the total mechanical energy (kinetic plus potential) is (a) less than zero, and (b) greater than zero? In each case, ignore air resistance and the gravitational effects of the sun, the moon, and the other planets.

Q13.14 Discuss whether this statement is correct: "In the absence of air resistance, the trajectory of a projectile thrown near the earth's surface is an *ellipse*, not a parabola."

Q13.15 The earth is closer to the sun in November than in May. In which of these months does it move faster in its orbit? Explain why.

Q13.16 A communications firm wants to place a satellite in orbit so that it is always directly above the earth's 45th parallel (latitude 45° north). This means that the plane of the orbit will not pass through the center of the earth. Is such an orbit possible? Why or why not?

Q13.17 At what point in an elliptical orbit is the acceleration maximum? At what point is it minimum? Justify your answers.

Q13.18 Which takes more fuel: a voyage from the earth to the moon or from the moon to the earth? Explain.

Q13.19 What would Kepler's third law be for circular orbits if an amendment to Newton's law of gravitation made the gravitational force inversely proportional to r^3? Would this change affect Kepler's other two laws? Explain.

Q13.20 In the elliptical orbit of Comet Halley shown in Fig. 13.20a, the sun's gravity is responsible for making the comet fall inward from aphelion to perihelion. But what is responsible for making the comet move from perihelion back outward to aphelion?

Q13.21 Many people believe that orbiting astronauts feel weightless because they are "beyond the pull of the earth's gravity." How far from the earth would a spacecraft have to travel to be truly beyond the earth's gravitational influence? If a spacecraft were really unaffected by the earth's gravity, would it remain in orbit? Explain. What is the real reason astronauts in orbit feel weightless?

Q13.22 As part of their training before going into orbit, astronauts ride in an airliner that is flown along the same parabolic trajectory as a freely falling projectile. Explain why this gives the same experience of apparent weightlessness as being in orbit.

EXERCISES

Section 13.1 Newton's Law of Gravitation

13.1 • What is the ratio of the gravitational pull of the sun on the moon to that of the earth on the moon? (Assume the distance of the moon from the sun can be approximated by the distance of the earth from the sun.) Use the data in Appendix F. Is it more accurate to say that the moon orbits the earth, or that the moon orbits the sun?

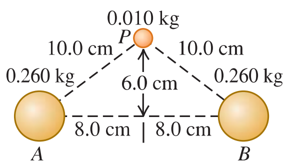
13.2 •• CP **Cavendish Experiment.** In the Cavendish balance apparatus shown in Fig. 13.4, suppose that $m_1 = 1.10$ kg, $m_2 = 25.0$ kg, and the rod connecting the m_1 pairs is 30.0 cm long. If, in each pair, m_1 and m_2 are 12.0 cm apart center to center, find (a) the net force and (b) the net torque (about the rotation axis) on the rotating part of the apparatus. (c) Does it seem that the torque in part (b) would be enough to easily rotate the rod? Suggest some ways to improve the sensitivity of this experiment.

13.3 • **Rendezvous in Space!** A couple of astronauts agree to rendezvous in space after hours. Their plan is to let gravity bring them together. One of them has a mass of 65 kg and the other a mass of 72 kg, and they start from rest 20.0 m apart. (a) Make a free-body diagram of each astronaut, and use it to find his or her initial acceleration. As a rough approximation, we can model the astronauts as uniform spheres. (b) If the astronauts' acceleration remained constant, how many days would they have to wait before reaching each other? (Careful! They *both* have acceleration toward each other.) (c) Would their acceleration, in fact, remain constant? If not, would it increase or decrease? Why?

13.4 •• Two uniform spheres, each with mass M and radius R, touch each other. What is the magnitude of their gravitational force of attraction?

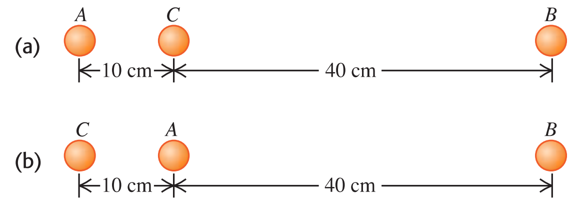
13.5 • Two uniform spheres, each of mass 0.260 kg, are fixed at points A and B (Fig. E13.5). Find the magnitude and direction of the initial acceleration of a uniform sphere with mass 0.010 kg if released from rest at point P and acted on only by forces of gravitational attraction of the spheres at A and B.

Figure **E13.5**



13.6 •• Find the magnitude and direction of the net gravitational force on mass A due to masses B and C in Fig. E13.6. Each mass is 2.00 kg.

Figure **E13.6**



13.7 • A typical adult human has a mass of about 70 kg. (a) What force does a full moon exert on such a human when it is directly overhead with its center 378,000 km away? (b) Compare this force with the force exerted on the human by the earth.

13.8 •• An 8.00-kg point mass and a 15.0-kg point mass are held in place 50.0 cm apart. A particle of mass m is released from a point between the two masses 20.0 cm from the 8.00-kg mass along the line connecting the two fixed masses. Find the magnitude and direction of the acceleration of the particle.

13.9 •• A particle of mass $3m$ is located 1.00 m from a particle of mass m. (a) Where should you put a third mass M so that the net gravitational force on M due to the two masses is exactly zero? (b) Is the equilibrium of M at this point stable or unstable (i) for points along the line connecting m and $3m$, and (ii) for points along the line passing through M and perpendicular to the line connecting m and $3m$?

13.10 •• The point masses m and $2m$ lie along the x-axis, with m at the origin and $2m$ at $x = L$. A third point mass M is moved along the x-axis. (a) At what point is the net gravitational force on M due to the other two masses equal to zero? (b) Sketch the x-component of the net force on M due to m and $2m$, taking quantities to the right as positive. Include the regions $x < 0$, $0 < x < L$, and $x > L$. Be especially careful to show the behavior of the graph on either side of $x = 0$ and $x = L$.

Section 13.2 Weight

13.11 •• At what distance above the surface of the earth is the acceleration due to the earth's gravity 0.980 m/s^2 if the acceleration due to gravity at the surface has magnitude 9.80 m/s^2?

13.12 • The mass of Venus is 81.5% that of the earth, and its radius is 94.9% that of the earth. (a) Compute the acceleration due to gravity on the surface of Venus from these data. (b) If a rock weighs 75.0 N on earth, what would it weigh at the surface of Venus?

13.13 • Titania, the largest moon of the planet Uranus, has $\frac{1}{8}$ the radius of the earth and $\frac{1}{1700}$ the mass of the earth. (a) What is the acceleration due to gravity at the surface of Titania? (b) What is the average density of Titania? (This is less than the density of rock, which is one piece of evidence that Titania is made primarily of ice.)

13.14 • Rhea, one of Saturn's moons, has a radius of 765 km and an acceleration due to gravity of 0.278 m/s^2 at its surface. Calculate its mass and average density.

13.15 •• Calculate the earth's gravity force on a 75-kg astronaut who is repairing the Hubble Space Telescope 600 km above the earth's surface, and then compare this value with his weight at the

earth's surface. In view of your result, explain why we say astronauts are weightless when they orbit the earth in a satellite such as a space shuttle. Is it because the gravitational pull of the earth is negligibly small?

Section 13.3 Gravitational Potential Energy

13.16 •• **Volcanoes on Io.** Jupiter's moon Io has active volcanoes (in fact, it is the most volcanically active body in the solar system) that eject material as high as 500 km (or even higher) above the surface. Io has a mass of 8.94×10^{22} kg and a radius of 1815 km. Ignore any variation in gravity over the 500-km range of the debris. How high would this material go on earth if it were ejected with the same speed as on Io?

13.17 • Use the results of Example 13.5 (Section 13.3) to calculate the escape speed for a spacecraft (a) from the surface of Mars and (b) from the surface of Jupiter. Use the data in Appendix F. (c) Why is the escape speed for a spacecraft independent of the spacecraft's mass?

13.18 •• Ten days after it was launched toward Mars in December 1998, the *Mars Climate Orbiter* spacecraft (mass 629 kg) was 2.87×10^6 km from the earth and traveling at 1.20×10^4 km/h relative to the earth. At this time, what were (a) the spacecraft's kinetic energy relative to the earth and (b) the potential energy of the earth–spacecraft system?

Section 13.4 The Motion of Satellites

13.19 • For a satellite to be in a circular orbit 780 km above the surface of the earth, (a) what orbital speed must it be given, and (b) what is the period of the orbit (in hours)?

13.20 •• **Aura Mission.** On July 15, 2004, NASA launched the Aura spacecraft to study the earth's climate and atmosphere. This satellite was injected into an orbit 705 km above the earth's surface. Assume a circular orbit. (a) How many hours does it take this satellite to make one orbit? (b) How fast (in km/s) is the Aura spacecraft moving?

13.21 •• Two satellites are in circular orbits around a planet that has radius 9.00×10^6 m. One satellite has mass 68.0 kg, orbital radius 5.00×10^7 m, and orbital speed 4800 m/s. The second satellite has mass 84.0 kg and orbital radius 3.00×10^7 m. What is the orbital speed of this second satellite?

13.22 •• **International Space Station.** The International Space Station makes 15.65 revolutions per day in its orbit around the earth. Assuming a circular orbit, how high is this satellite above the surface of the earth?

13.23 • Deimos, a moon of Mars, is about 12 km in diameter with mass 2.0×10^{15} kg. Suppose you are stranded alone on Deimos and want to play a one-person game of baseball. You would be the pitcher, and you would be the batter! (a) With what speed would you have to throw a baseball so that it would go into a circular orbit just above the surface and return to you so you could hit it? Do you think you could actually throw it at this speed? (b) How long (in hours) after throwing the ball should you be ready to hit it? Would this be an action-packed baseball game?

Section 13.5 Kepler's Laws and the Motion of Planets

13.24 •• **Planet Vulcan.** Suppose that a planet were discovered between the sun and Mercury, with a circular orbit of radius equal to $\frac{2}{3}$ of the average orbit radius of Mercury. What would be the orbital period of such a planet? (Such a planet was once postulated, in part to explain the precession of Mercury's orbit. It was even given the name Vulcan, although we now have no evidence that it actually exists. Mercury's precession has been explained by general relativity.)

13.25 •• The star Rho^1 Cancri is 57 light-years from the earth and has a mass 0.85 times that of our sun. A planet has been detected in a circular orbit around Rho^1 Cancri with an orbital radius equal to 0.11 times the radius of the earth's orbit around the sun. What are (a) the orbital speed and (b) the orbital period of the planet of Rho^1 Cancri?

13.26 •• In March 2006, two small satellites were discovered orbiting Pluto, one at a distance of 48,000 km and the other at 64,000 km. Pluto already was known to have a large satellite Charon, orbiting at 19,600 km with an orbital period of 6.39 days. Assuming that the satellites do not affect each other, find the orbital periods of the two small satellites *without* using the mass of Pluto.

13.27 • (a) Use Fig. 13.18 to show that the sun–planet distance at perihelion is $(1 - e)a$, the sun–planet distance at aphelion is $(1 + e)a$, and therefore the sum of these two distances is $2a$. (b) When the dwarf planet Pluto was at perihelion in 1989, it was almost 100 million km closer to the sun than Neptune. The semi-major axes of the orbits of Pluto and Neptune are 5.92×10^{12} m and 4.50×10^{12} m, respectively, and the eccentricities are 0.248 and 0.010. Find Pluto's closest distance and Neptune's farthest distance from the sun. (c) How many years after being at perihelion in 1989 will Pluto again be at perihelion?

13.28 •• **Hot Jupiters.** In 2004 astronomers reported the discovery of a large Jupiter-sized planet orbiting very close to the star HD 179949 (hence the term "hot Jupiter"). The orbit was just $\frac{1}{9}$ the distance of Mercury from our sun, and it takes the planet only 3.09 days to make one orbit (assumed to be circular). (a) What is the mass of the star? Express your answer in kilograms and as a multiple of our sun's mass. (b) How fast (in km/s) is this planet moving?

13.29 •• **Planets Beyond the Solar System.** On October 15, 2001, a planet was discovered orbiting around the star HD 68988. Its orbital distance was measured to be 10.5 million kilometers from the center of the star, and its orbital period was estimated at 6.3 days. What is the mass of HD 68988? Express your answer in kilograms and in terms of our sun's mass. (Consult Appendix F.)

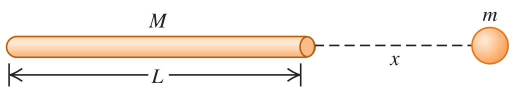
Section 13.6 Spherical Mass Distributions

13.30 • A uniform, spherical, 1000.0-kg shell has a radius of 5.00 m. (a) Find the gravitational force this shell exerts on a 2.00-kg point mass placed at the following distances from the center of the shell: (i) 5.01 m, (ii) 4.99 m, (iii) 2.72 m. (b) Sketch a qualitative graph of the magnitude of the gravitational force this sphere exerts on a point mass m as a function of the distance r of m from the center of the sphere. Include the region from $r = 0$ to $r \to \infty$.

13.31 •• A uniform, solid, 1000.0-kg sphere has a radius of 5.00 m. (a) Find the gravitational force this sphere exerts on a 2.00-kg point mass placed at the following distances from the center of the sphere: (i) 5.01 m, (ii) 2.50 m. (b) Sketch a qualitative graph of the magnitude of the gravitational force this sphere exerts on a point mass m as a function of the distance r of m from the center of the sphere. Include the region from $r = 0$ to $r \to \infty$.

13.32 • **CALC** A thin, uniform rod has length L and mass M. A small uniform sphere of mass m is placed a distance x from one end of the rod, along the axis of the rod (Fig. E13.32). (a) Calculate

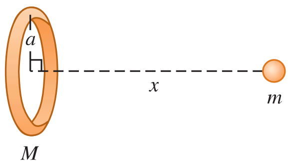
Figure **E13.32**



the gravitational potential energy of the rod–sphere system. Take the potential energy to be zero when the rod and sphere are infinitely far apart. Show that your answer reduces to the expected result when x is much larger than L. (*Hint:* Use the power series expansion for $\ln(1 + x)$ given in Appendix B.) (b) Use $F_x = -dU/dx$ to find the magnitude and direction of the gravitational force exerted on the sphere by the rod (see Section 7.4). Show that your answer reduces to the expected result when x is much larger than L.

13.33 • **CALC** Consider the ring-shaped body of Fig. E13.33. A particle with mass m is placed a distance x from the center of the ring, along the line through the center of the ring and perpendicular to its plane. (a) Calculate the gravitational potential energy U of this system. Take the potential energy to be zero when the two objects are far apart. (b) Show that your answer to part (a) reduces to the expected result when x is much larger than the radius a of the ring. (c) Use $F_x = -dU/dx$ to find the magnitude and direction of the force on the particle (see Section 7.4). (d) Show that your answer to part (c) reduces to the expected result when x is much larger than a. (e) What are the values of U and F_x when $x = 0$? Explain why these results make sense.

Figure **E13.33**



Section 13.7 Apparent Weight and the Earth's Rotation

13.34 •• **A Visit to Santa.** You decide to visit Santa Claus at the north pole to put in a good word about your splendid behavior throughout the year. While there, you notice that the elf Sneezy, when hanging from a rope, produces a tension of 475.0 N in the rope. If Sneezy hangs from a similar rope while delivering presents at the earth's equator, what will the tension in it be? (Recall that the earth is rotating about an axis through its north and south poles.) Consult Appendix F and start with a free-body diagram of Sneezy at the equator.

13.35 • The acceleration due to gravity at the north pole of Neptune is approximately 10.7 m/s^2. Neptune has mass 1.0×10^{26} kg and radius 2.5×10^4 km and rotates once around its axis in about 16 h. (a) What is the gravitational force on a 5.0-kg object at the north pole of Neptune? (b) What is the apparent weight of this same object at Neptune's equator? (Note that Neptune's "surface" is gaseous, not solid, so it is impossible to stand on it.)

Section 13.8 Black Holes

13.36 •• **Mini Black Holes.** Cosmologists have speculated that black holes the size of a proton could have formed during the early days of the Big Bang when the universe began. If we take the diameter of a proton to be 1.0×10^{-15} m, what would be the mass of a mini black hole?

13.37 •• **At the Galaxy's Core.** Astronomers have observed a small, massive object at the center of our Milky Way galaxy (see Section 13.8). A ring of material orbits this massive object; the ring has a diameter of about 15 light-years and an orbital speed of about 200 km/s. (a) Determine the mass of the object at the center of the Milky Way galaxy. Give your answer both in kilograms and in solar masses (one solar mass is the mass of the sun). (b) Observations of stars, as well as theories of the structure of stars, suggest that it is impossible for a single star to have a mass of more than about 50 solar masses. Can this massive object be a single, ordinary star? (c) Many astronomers believe that the massive object at the center of the Milky Way galaxy is a black hole. If so, what must the Schwarzschild radius of this black hole be? Would a black hole of this size fit inside the earth's orbit around the sun?

13.38 • (a) Show that a black hole attracts an object of mass m with a force of $mc^2 R_S/(2r^2)$, where r is the distance between the object and the center of the black hole. (b) Calculate the magnitude of the gravitational force exerted by a black hole of Schwarzschild radius 14.0 mm on a 5.00-kg mass 3000 km from it. (c) What is the mass of this black hole?

13.39 • In 2005 astronomers announced the discovery of a large black hole in the galaxy Markarian 766 having clumps of matter orbiting around once every 27 hours and moving at 30,000 km/s. (a) How far are these clumps from the center of the black hole? (b) What is the mass of this black hole, assuming circular orbits? Express your answer in kilograms and as a multiple of our sun's mass. (c) What is the radius of its event horizon?

PROBLEMS

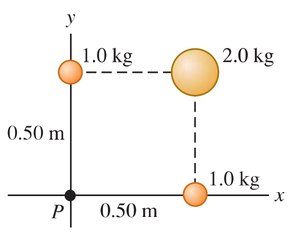
13.40 ••• Four identical masses of 800 kg each are placed at the corners of a square whose side length is 10.0 cm. What is the net gravitational force (magnitude and direction) on one of the masses, due to the other three?

13.41 ••• Neutron stars, such as the one at the center of the Crab Nebula, have about the same mass as our sun but have a *much* smaller diameter. If you weigh 675 N on the earth, what would you weigh at the surface of a neutron star that has the same mass as our sun and a diameter of 20 km?

13.42 ••• **CP** **Exploring Europa.** There is strong evidence that Europa, a satellite of Jupiter, has a liquid ocean beneath its icy surface. Many scientists think we should land a vehicle there to search for life. Before launching it, we would want to test such a lander under the gravity conditions at the surface of Europa. One way to do this is to put the lander at the end of a rotating arm in an orbiting earth satellite. If the arm is 4.25 m long and pivots about one end, at what angular speed (in rpm) should it spin so that the acceleration of the lander is the same as the acceleration due to gravity at the surface of Europa? The mass of Europa is 4.8×10^{22} kg and its diameter is 3138 km.

13.43 • Three uniform spheres are fixed at the positions shown in Fig. P13.43. (a) What are the magnitude and direction of the force on a 0.0150-kg particle placed at P? (b) If the spheres are in deep outer space and a 0.0150-kg particle is released from rest 300 m from the origin along a line 45° below the $-x$-axis, what will the particle's speed be when it reaches the origin?

Figure **P13.43**



13.44 •• A uniform sphere with mass 60.0 kg is held with its center at the origin, and a second uniform sphere with mass 80.0 kg is held with its center at the point $x = 0$, $y = 3.00$ m. (a) What are the magnitude and direction of the net gravitational force due to these objects on a third uniform sphere with mass 0.500 kg placed at the point $x = 4.00$ m, $y = 0$? (b) Where, other than infinitely far away, could the third sphere be placed such that the net gravitational force acting on it from the other two spheres is equal to zero?

13.45 •• CP BIO Hip Wear on the Moon. (a) Use data from Appendix F to calculate the acceleration due to gravity on the moon. (b) Calculate the friction force on a walking 65-kg astronaut carrying a 43-kg instrument pack on the moon if the coefficient of kinetic friction at her hip joint is 0.0050. (c) What would be the friction force on earth for this astronaut?

13.46 •• Mission to Titan. On December 25, 2004, the *Huygens* probe separated from the *Cassini* spacecraft orbiting Saturn and began a 22-day journey to Saturn's giant moon Titan, on whose surface it landed. Besides the data in Appendix F, it is useful to know that Titan is 1.22×10^6 km from the center of Saturn and has a mass of 1.35×10^{23} kg and a diameter of 5150 km. At what distance from Titan should the gravitational pull of Titan just balance the gravitational pull of Saturn?

13.47 •• The asteroid Toro has a radius of about 5.0 km. Consult Appendix F as necessary. (a) Assuming that the density of Toro is the same as that of the earth ($5.5\ \text{g/cm}^3$), find its total mass and find the acceleration due to gravity at its surface. (b) Suppose an object is to be placed in a circular orbit around Toro, with a radius just slightly larger than the asteroid's radius. What is the speed of the object? Could you launch yourself into orbit around Toro by running?

13.48 ••• At a certain instant, the earth, the moon, and a stationary 1250-kg spacecraft lie at the vertices of an equilateral triangle whose sides are 3.84×10^5 km in length. (a) Find the magnitude and direction of the net gravitational force exerted on the spacecraft by the earth and moon. State the direction as an angle measured from a line connecting the earth and the spacecraft. In a sketch, show the earth, the moon, the spacecraft, and the force vector. (b) What is the minimum amount of work that you would have to do to move the spacecraft to a point far from the earth and moon? You can ignore any gravitational effects due to the other planets or the sun.

13.49 ••• CP An experiment is performed in deep space with two uniform spheres, one with mass 50.0 kg and the other with mass 100.0 kg. They have equal radii, $r = 0.20$ m.The spheres are released from rest with their centers 40.0 m apart. They accelerate toward each other because of their mutual gravitational attraction. You can ignore all gravitational forces other than that between the two spheres. (a) Explain why linear momentum is conserved. (b) When their centers are 20.0 m apart, find (i) the speed of each sphere and (ii) the magnitude of the relative velocity with which one sphere is approaching the other. (c) How far from the initial position of the center of the 50.0-kg sphere do the surfaces of the two spheres collide?

13.50 •• CP Submarines on Europa. Some scientists are eager to send a remote-controlled submarine to Jupiter's moon Europa to search for life in its oceans below an icy crust. Europa's mass has been measured to be 4.8×10^{22} kg, its diameter is 3138 km, and it has no appreciable atmosphere. Assume that the layer of ice at the surface is not thick enough to exert substantial force on the water. If the windows of the submarine you are designing are 25.0 cm square and can stand a maximum inward force of 9750 N per window, what is the greatest depth to which this submarine can safely dive?

13.51 • Geosynchronous Satellites. Many satellites are moving in a circle in the earth's equatorial plane. They are at such a height above the earth's surface that they always remain above the same point. (a) Find the altitude of these satellites above the earth's surface. (Such an orbit is said to be *geosynchronous*.) (b) Explain, with a sketch, why the radio signals from these satellites cannot directly reach receivers on earth that are north of 81.3° N latitude.

13.52 ••• A landing craft with mass 12,500 kg is in a circular orbit 5.75×10^5 m above the surface of a planet. The period of the orbit is 5800 s. The astronauts in the lander measure the diameter of the planet to be 9.60×10^6 m. The lander sets down at the north pole of the planet. What is the weight of an 85.6-kg astronaut as he steps out onto the planet's surface?

13.53 ••• What is the escape speed from a 300-km-diameter asteroid with a density of $2500\ \text{kg/m}^3$?

13.54 •• (a) Asteroids have average densities of about $2500\ \text{kg/m}^3$ and radii from 470 km down to less than a kilometer. Assuming that the asteroid has a spherically symmetric mass distribution, estimate the radius of the largest asteroid from which you could escape simply by jumping off. (*Hint:* You can estimate your jump speed by relating it to the maximum height that you can jump on earth.) (b) Europa, one of Jupiter's four large moons, has a radius of 1570 km. The acceleration due to gravity at its surface is $1.33\ \text{m/s}^2$. Calculate its average density.

13.55 ••• (a) Suppose you are at the earth's equator and observe a satellite passing directly overhead and moving from west to east in the sky. Exactly 12.0 hours later, you again observe this satellite to be directly overhead. How far above the earth's surface is the satellite's orbit? (b) You observe another satellite directly overhead and traveling east to west. This satellite is again overhead in 12.0 hours. How far is this satellite's orbit above the surface of the earth?

13.56 •• Planet X rotates in the same manner as the earth, around an axis through its north and south poles, and is perfectly spherical. An astronaut who weighs 943.0 N on the earth weighs 915.0 N at the north pole of Planet X and only 850.0 N at its equator. The distance from the north pole to the equator is 18,850 km, measured along the surface of Planet X. (a) How long is the day on Planet X? (b) If a 45,000-kg satellite is placed in a circular orbit 2000 km above the surface of Planet X, what will be its orbital period?

13.57 •• There are two equations from which a change in the gravitational potential energy U of the system of a mass m and the earth can be calculated. One is $U = mgy$ (Eq. 7.2). The other is $U = -Gm_E m/r$ (Eq. 13.9). As shown in Section 13.3, the first equation is correct only if the gravitational force is a constant over the change in height Δy. The second is always correct. Actually, the gravitational force is never exactly constant over any change in height, but if the variation is small, we can ignore it. Consider the difference in U between a mass at the earth's surface and a distance h above it using both equations, and find the value of h for which Eq. (7.2) is in error by 1%. Express this value of h as a fraction of the earth's radius, and also obtain a numerical value for it.

13.58 ••• CP Your starship, the *Aimless Wanderer,* lands on the mysterious planet Mongo. As chief scientist-engineer, you make the following measurements: A 2.50-kg stone thrown upward from the ground at 12.0 m/s returns to the ground in 6.00 s; the circumference of Mongo at the equator is 2.00×10^5 km; and there is no appreciable atmosphere on Mongo. The starship commander, Captain Confusion, asks for the following information: (a) What is the mass of Mongo? (b) If the *Aimless Wanderer* goes into a circular orbit 30,000 km above the surface of Mongo, how many hours will it take the ship to complete one orbit?

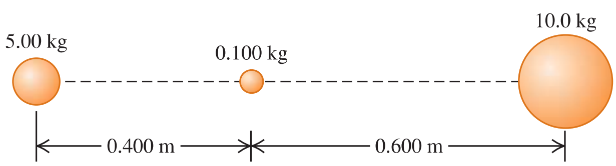
13.59 •• CP An astronaut, whose mission is to go where no one has gone before, lands on a spherical planet in a distant galaxy. As she stands on the surface of the planet, she releases a small rock from rest and finds that it takes the rock 0.480 s to fall 1.90 m. If the radius of the planet is 8.60×10^7 m, what is the mass of the planet?

13.60 •• In Example 13.5 (Section 13.3) we ignored the gravitational effects of the moon on a spacecraft en route from the earth to the moon. In fact, we must include the gravitational potential energy due to the moon as well. For this problem, you can ignore the motion of the earth and moon. (a) If the moon has radius R_M and the distance between the centers of the earth and the moon is R_{EM}, find the total gravitational potential energy of the particle–earth and particle–moon systems when a particle with mass m is between the earth and the moon, and a distance r from the center of the earth. Take the gravitational potential energy to be zero when the objects are far from each other. (b) There is a point along a line between the earth and the moon where the net gravitational force is zero. Use the expression derived in part (a) and numerical values from Appendix F to find the distance of this point from the center of the earth. With what speed must a spacecraft be launched from the surface of the earth just barely to reach this point? (c) If a spacecraft were launched from the earth's surface toward the moon with an initial speed of 11.2 km/s, with what speed would it impact the moon?

13.61 •• Calculate the percent difference between your weight in Sacramento, near sea level, and at the top of Mount Everest, which is 8800 m above sea level.

13.62 •• The 0.100-kg sphere in Fig. P13.62 is released from rest at the position shown in the sketch, with its center 0.400 m from the center of the 5.00-kg mass. Assume that the only forces on the 0.100-kg sphere are the gravitational forces exerted by the other two spheres and that the 5.00-kg and 10.0-kg spheres are held in place at their initial positions. What is the speed of the 0.100-kg sphere when it has moved 0.400 m to the right from its initial position?

Figure **P13.62**



13.63 ••• An unmanned spacecraft is in a circular orbit around the moon, observing the lunar surface from an altitude of 50.0 km (see Appendix F). To the dismay of scientists on earth, an electrical fault causes an on-board thruster to fire, decreasing the speed of the spacecraft by 20.0 m/s. If nothing is done to correct its orbit, with what speed (in km/h) will the spacecraft crash into the lunar surface?

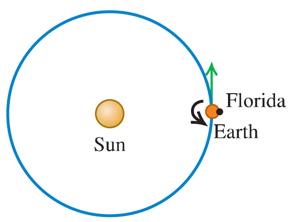
13.64 ••• Mass of a Comet. On July 4, 2005, the NASA spacecraft Deep Impact fired a projectile onto the surface of Comet Tempel 1. This comet is about 9.0 km across. Observations of surface debris released by the impact showed that dust with a speed as low as 1.0 m/s was able to escape the comet. (a) Assuming a spherical shape, what is the mass of this comet? (*Hint:* See Example 13.5 in Section 13.3.) (b) How far from the comet's center will this debris be when it has lost (i) 90.0% of its initial kinetic energy at the surface and (ii) all of its kinetic energy at the surface?

13.65 • Falling Hammer. A hammer with mass m is dropped from rest from a height h above the earth's surface. This height is not necessarily small compared with the radius R_E of the earth. If you ignore air resistance, derive an expression for the speed v of the hammer when it reaches the surface of the earth. Your expression should involve h, R_E, and m_E, the mass of the earth.

13.66 • (a) Calculate how much work is required to launch a spacecraft of mass m from the surface of the earth (mass m_E, radius R_E) and place it in a circular *low earth orbit*—that is, an orbit whose altitude above the earth's surface is much less than R_E. (As an example, the International Space Station is in low earth orbit at an altitude of about 400 km, much less than $R_E = 6380$ km.) You can ignore the kinetic energy that the spacecraft has on the ground due to the earth's rotation. (b) Calculate the minimum amount of additional work required to move the spacecraft from low earth orbit to a very great distance from the earth. You can ignore the gravitational effects of the sun, the moon, and the other planets. (c) Justify the statement: "In terms of energy, low earth orbit is halfway to the edge of the universe."

13.67 • A spacecraft is to be launched from the surface of the earth so that it will escape from the solar system altogether. (a) Find the speed relative to the center of the earth with which the spacecraft must be launched. Take into consideration the gravitational effects of both the earth and the sun, and include the effects of the earth's orbital speed, but ignore air resistance. (b) The rotation of the earth can help this spacecraft achieve escape speed. Find the speed that the spacecraft must have relative to the earth's *surface* if the spacecraft is launched from Florida at the point shown in Fig. P13.67. The rotation and orbital motions of the earth are in the same direction. The launch facilities in Florida are 28.5° north of the equator. (c) The European Space Agency (ESA) uses launch facilities in French Guiana (immediately north of Brazil), 5.15° north of the equator. What speed relative to the earth's surface would a spacecraft need to escape the solar system if launched from French Guiana?

Figure **P13.67**



13.68 • Gravity Inside the Earth. Find the gravitational force that the earth exerts on a 10.0-kg mass if it is placed at the following locations. Consult Fig. 13.9, and assume a constant density through each of the interior regions (mantle, outer core, inner core), but *not* the same density in each of these regions. Use the graph to estimate the average density for each region: (a) at the surface of the earth; (b) at the outer surface of the molten outer core; (c) at the surface of the solid inner core; (d) at the center of the earth.

13.69 • Kirkwood Gaps. Hundreds of thousands of asteroids orbit the sun within the *asteroid belt,* which extends from about 3×10^8 km to about 5×10^8 km from the sun. (a) Find the orbital period (in years) of (i) an asteroid at the inside of the belt and (ii) an asteroid at the outside of the belt. Assume circular orbits. (b) In 1867 the American astronomer Daniel Kirkwood pointed out that several gaps exist in the asteroid belt where relatively few asteroids are found. It is now understood that these *Kirkwood gaps* are caused by the gravitational attraction of Jupiter, the largest planet, which orbits the sun once every 11.86 years. As an example, if an asteroid has an orbital period half that of Jupiter, or 5.93 years, on every other orbit this asteroid would be at its closest to Jupiter and feel a strong attraction toward the planet. This attraction, acting over and over on successive orbits, could sweep asteroids out of the Kirkwood gap. Use this hypothesis to determine the orbital radius for this Kirkwood gap. (c) One of several other Kirkwood gaps appears at a distance from the sun where the orbital period is 0.400 that of Jupiter. Explain why this happens, and find the orbital radius for this Kirkwood gap.

13.70 ••• If a satellite is in a sufficiently low orbit, it will encounter air drag from the earth's atmosphere. Since air drag does negative work (the force of air drag is directed opposite the motion), the mechanical energy will decrease. According to Eq. (13.13), if E decreases (becomes more negative), the radius r of the orbit will decrease. If air drag is relatively small, the satellite can be considered to be in a circular orbit of continually decreasing radius. (a) According to Eq. (13.10), if the radius of a satellite's circular orbit decreases, the satellite's orbital speed v *increases.* How can you reconcile this with the statement that the mechanical energy *decreases?* (*Hint:* Is air drag the only force that does work on the satellite as the orbital radius decreases?) (b) Due to air drag, the radius of a satellite's circular orbit decreases from r to $r - \Delta r$, where the positive quantity Δr is much less than r. The mass of the satellite is m. Show that the increase in orbital speed is $\Delta v = +(\Delta r/2)\sqrt{Gm_E/r^3}$; that the change in kinetic energy is $\Delta K = +(Gm_E\,m/2r^2)\,\Delta r$; that the change in gravitational potential energy is $\Delta U = -2\,\Delta K = -(Gm_E\,m/r^2)\,\Delta r$; and that the amount of work done by the force of air drag is $W = -(Gm_E\,m/2r^2)\,\Delta r$. Interpret these results in light of your comments in part (a). (c) A satellite with mass 3000 kg is initially in a circular orbit 300 km above the earth's surface. Due to air drag, the satellite's altitude decreases to 250 km. Calculate the initial orbital speed; the increase in orbital speed; the initial mechanical energy; the change in kinetic energy; the change in gravitational potential energy; the change in mechanical energy; and the work done by the force of air drag. (d) Eventually a satellite will descend to a low enough altitude in the atmosphere that the satellite burns up and the debris falls to the earth. What becomes of the initial mechanical energy?

13.71 • Binary Star—Equal Masses. Two identical stars with mass M orbit around their center of mass. Each orbit is circular and has radius R, so that the two stars are always on opposite sides of the circle. (a) Find the gravitational force of one star on the other. (b) Find the orbital speed of each star and the period of the orbit. (c) How much energy would be required to separate the two stars to infinity?

13.72 •• CP Binary Star—Different Masses. Two stars, with masses M_1 and M_2, are in circular orbits around their center of mass. The star with mass M_1 has an orbit of radius R_1; the star with mass M_2 has an orbit of radius R_2. (a) Show that the ratio of the orbital radii of the two stars equals the reciprocal of the ratio of their masses—that is, $R_1/R_2 = M_2/M_1$. (b) Explain why the two stars have the same orbital period, and show that the period T is given by $T = 2\pi(R_1 + R_2)^{3/2}/\sqrt{G(M_1 + M_2)}$. (c) The two stars in a certain binary star system move in circular orbits. The first star, Alpha, has an orbital speed of 36.0 km/s. The second star, Beta, has an orbital speed of 12.0 km/s. The orbital period is 137 d. What are the masses of each of the two stars? (d) One of the best candidates for a black hole is found in the binary system called A0620-0090. The two objects in the binary system are an orange star, V616 Monocerotis, and a compact object believed to be a black hole (see Fig. 13.27). The orbital period of A0620-0090 is 7.75 hours, the mass of V616 Monocerotis is estimated to be 0.67 times the mass of the sun, and the mass of the black hole is estimated to be 3.8 times the mass of the sun. Assuming that the orbits are circular, find the radius of each object's orbit and the orbital speed of each object. Compare these answers to the orbital radius and orbital speed of the earth in its orbit around the sun.

13.73 ••• Comets travel around the sun in elliptical orbits with large eccentricities. If a comet has speed 2.0×10^4 m/s when at a distance of 2.5×10^{11} m from the center of the sun, what is its speed when at a distance of 5.0×10^{10} m?

13.74 •• CP An astronaut is standing at the north pole of a newly discovered, spherically symmetric planet of radius R. In his hands he holds a container full of a liquid with mass m and volume V. At the surface of the liquid, the pressure is p_0; at a depth d below the surface, the pressure has a greater value p. From this information, determine the mass of the planet.

13.75 •• CALC The earth does not have a uniform density; it is most dense at its center and least dense at its surface. An approximation of its density is $\rho(r) = A - Br$, where $A = 12{,}700\ \text{kg/m}^3$ and $B = 1.50 \times 10^{-3}\ \text{kg/m}^4$. Use $R = 6.37 \times 10^6$ m for the radius of the earth approximated as a sphere. (a) Geological evidence indicates that the densities are 13,100 kg/m^3 and 2400 kg/m^3 at the earth's center and surface, respectively. What values does the linear approximation model give for the densities at these two locations? (b) Imagine dividing the earth into concentric, spherical shells. Each shell has radius r, thickness dr, volume $dV = 4\pi r^2\,dr$, and mass $dm = \rho(r)dV$. By integrating from $r = 0$ to $r = R$, show that the mass of the earth in this model is $M = \frac{4}{3}\pi R^3(A - \frac{3}{4}BR)$. (c) Show that the given values of A and B give the correct mass of the earth to within 0.4%. (d) We saw in Section 13.6 that a uniform spherical shell gives no contribution to g inside it. Show that $g(r) = \frac{4}{3}\pi Gr(A - \frac{3}{4}Br)$ inside the earth in this model. (e) Verify that the expression of part (d) gives $g = 0$ at the center of the earth and $g = 9.85\ \text{m/s}^2$ at the surface. (f) Show that in this model g does *not* decrease uniformly with depth but rather has a maximum of $4\pi GA^2/9B = 10.01\ \text{m/s}^2$ at $r = 2A/3B = 5640$ km.

13.76 •• CP CALC In Example 13.10 (Section 13.6) we saw that inside a planet of uniform density (not a realistic assumption for the earth) the acceleration due to gravity increases uniformly with distance from the center of the planet. That is, $g(r) = g_s r/R$, where g_s is the acceleration due to gravity at the surface, r is the distance from the center of the planet, and R is the radius of the planet. The interior of the planet can be treated approximately as an incompressible fluid of density ρ. (a) Replace the height y in Eq. (12.4) with the radial coordinate r and integrate to find the pressure inside a uniform planet as a function of r. Let the pressure at the surface be zero. (This means ignoring the pressure of the planet's atmosphere.) (b) Using this model, calculate the pressure at the center of the earth. (Use a value of ρ equal to the average density of the earth, calculated from the mass and radius given in Appendix F.) (c) Geologists estimate the pressure at the center of the earth to be approximately 4×10^{11} Pa. Does this agree with your calculation for the pressure at $r = 0$? What might account for any differences?

13.77 ••• CP Consider a spacecraft in an elliptical orbit around the earth. At the low point, or perigee, of its orbit, it is 400 km above the earth's surface; at the high point, or apogee, it is 4000 km above the earth's surface. (a) What is the period of the spacecraft's orbit? (b) Using conservation of angular momentum, find the ratio of the spacecraft's speed at perigee to its speed at apogee. (c) Using conservation of energy, find the speed at perigee and the speed at apogee. (d) It is necessary to have the spacecraft escape from the earth completely. If the spacecraft's rockets are fired at perigee, by how much would the speed have to be increased to achieve this? What if the rockets were fired at apogee? Which point in the orbit is more efficient to use?

13.78 • The planet Uranus has a radius of 25,560 km and a surface acceleration due to gravity of 11.1 m/s^2 at its poles. Its moon Miranda (discovered by Kuiper in 1948) is in a circular orbit about Uranus at an altitude of 104,000 km above the planet's surface. Miranda has a mass of 6.6×10^{19} kg and a radius of 235 km. (a) Calculate the mass of Uranus from the given data. (b) Calculate

the magnitude of Miranda's acceleration due to its orbital motion about Uranus. (c) Calculate the acceleration due to Miranda's gravity at the surface of Miranda. (d) Do the answers to parts (b) and (c) mean that an object released 1 m above Miranda's surface on the side toward Uranus will fall *up* relative to Miranda? Explain.

13.79 ••• A 5000-kg spacecraft is in a circular orbit 2000 km above the surface of Mars. How much work must the spacecraft engines perform to move the spacecraft to a circular orbit that is 4000 km above the surface?

13.80 •• One of the brightest comets of the 20th century was Comet Hyakutake, which passed close to the sun in early 1996. The orbital period of this comet is estimated to be about 30,000 years. Find the semi-major axis of this comet's orbit. Compare it to the average sun–Pluto distance and to the distance to Alpha Centauri, the nearest star to the sun, which is 4.3 light-years distant.

13.81 ••• **CALC** Planets are not uniform inside. Normally, they are densest at the center and have decreasing density outward toward the surface. Model a spherically symmetric planet, with the same radius as the earth, as having a density that decreases linearly with distance from the center. Let the density be $15.0 \times 10^3\ \mathrm{kg/m^3}$ at the center and $2.0 \times 10^3\ \mathrm{kg/m^3}$ at the surface. What is the acceleration due to gravity at the surface of this planet?

13.82 •• **CALC** A uniform wire with mass M and length L is bent into a semicircle. Find the magnitude and direction of the gravitational force this wire exerts on a point with mass m placed at the center of curvature of the semicircle.

13.83 ••• **CALC** An object in the shape of a thin ring has radius a and mass M. A uniform sphere with mass m and radius R is placed with its center at a distance x to the right of the center of the ring, along a line through the center of the ring, and perpendicular to its plane (see Fig. E13.33). What is the gravitational force that the sphere exerts on the ring-shaped object? Show that your result reduces to the expected result when x is much larger than a.

13.84 ••• **CALC** A thin, uniform rod has length L and mass M. Calculate the magnitude of the gravitational force the rod exerts on a particle with mass m that is at a point along the axis of the rod a distance x from one end (see Fig. E13.32). Show that your result reduces to the expected result when x is much larger than L.

13.85 • **CALC** A shaft is drilled from the surface to the center of the earth (see Fig. 13.24). As in Example 13.10 (Section 13.6), make the unrealistic assumption that the density of the earth is uniform. With this approximation, the gravitational force on an object with mass m, that is inside the earth at a distance r from the center, has magnitude $F_g = Gm_E mr/R_E^3$ (as shown in Example 13.10) and points toward the center of the earth. (a) Derive an expression for the gravitational potential energy $U(r)$ of the object–earth system as a function of the object's distance from the center of the earth. Take the potential energy to be zero when the object is at the center of the earth. (b) If an object is released in the shaft at the earth's surface, what speed will it have when it reaches the center of the earth?

CHALLENGE PROBLEMS

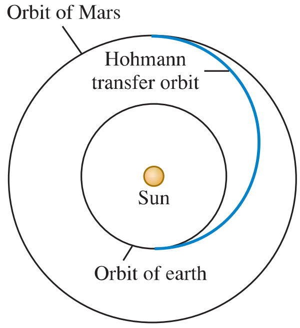
13.86 ••• (a) When an object is in a circular orbit of radius r around the earth (mass m_E), the period of the orbit is T, given by Eq. (13.12), and the orbital speed is v, given by Eq. (13.10). Show that when the object is moved into a circular orbit of slightly larger radius $r + \Delta r$, where $\Delta r \ll r$, its new period is $T + \Delta T$ and its new orbital speed is $v - \Delta v$, where Δr, ΔT, and Δv are all positive quantities and

$$\Delta T = \frac{3\pi\,\Delta r}{v} \qquad \text{and} \qquad \Delta v = \frac{\pi\,\Delta r}{T}$$

[*Hint:* Use the expression $(1 + x)^n \approx 1 + nx$, valid for $|x| \ll 1$.] (b) The International Space Station (ISS) is in a nearly circular orbit at an altitude of 398.00 km above the surface of the earth. A maintenance crew is about to arrive on the space shuttle that is also in a circular orbit in the same orbital plane as the ISS, but with an altitude of 398.10 km. The crew has come to remove a faulty 125-m electrical cable, one end of which is attached to the ISS and the other end of which is floating free in space. The plan is for the shuttle to snag the free end just at the moment that the shuttle, the ISS, and the center of the earth all lie along the same line. The cable will then break free from the ISS when it becomes taut. How long after the free end is caught by the space shuttle will it detach from the ISS? Give your answer in minutes. (c) If the shuttle misses catching the cable, show that the crew must wait a time $t \approx T^2/\Delta T$ before they have a second chance. Find the numerical value of t and explain whether it would be worth the wait.

13.87 ••• **Interplanetary Navigation.** The most efficient way to send a spacecraft from the earth to another planet is by using a *Hohmann transfer orbit* (Fig. P13.87). If the orbits of the departure and destination planets are circular, the Hohmann transfer orbit is an elliptical orbit whose perihelion and aphelion are tangent to the orbits of the two planets. The rockets are fired briefly at the departure planet to put the spacecraft into the transfer orbit; the spacecraft then coasts until it reaches the destination planet. The rockets are then fired again to put the spacecraft into the same orbit about the sun as the destination planet. (a) For a flight from earth to Mars, in what direction must the rockets be fired at the earth and at Mars: in the direction of motion, or opposite the direction of motion? What about for a flight from Mars to the earth? (b) How long does a one-way trip from the the earth to Mars take, between the firings of the rockets? (c) To reach Mars from the earth, the launch must be timed so that Mars will be at the right spot when the spacecraft reaches Mars's orbit around the sun. At launch, what must the angle between a sun–Mars line and a sun–earth line be? Use data from Appendix F.

Figure **P13.87**

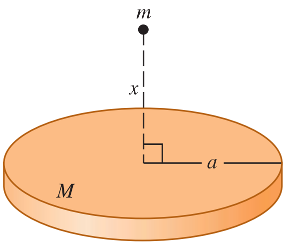


13.88 ••• **CP Tidal Forces near a Black Hole.** An astronaut inside a spacecraft, which protects her from harmful radiation, is orbiting a black hole at a distance of 120 km from its center. The black hole is 5.00 times the mass of the sun and has a Schwarzschild radius of 15.0 km. The astronaut is positioned inside the spaceship such that one of her 0.030-kg ears is 6.0 cm farther from the black hole than the center of mass of the spacecraft and the other ear is 6.0 cm closer. (a) What is the tension between her ears? Would the astronaut find it difficult to keep from being torn apart by the gravitational forces? (Since her whole body orbits with the same angular velocity, one ear is moving too slowly for the radius of its orbit and the other is moving too fast. Hence her head must exert forces on her

ears to keep them in their orbits.) (b) Is the center of gravity of her head at the same point as the center of mass? Explain.

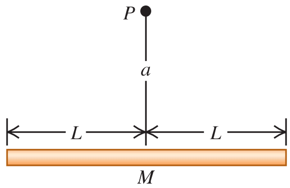
13.89 ••• **CALC** Mass M is distributed uniformly over a disk of radius a. Find the gravitational force (magnitude and direction) between this disk-shaped mass and a particle with mass m located a distance x above the center of the disk (Fig. P13.89). Does your result reduce to the correct expression as x becomes very large? (*Hint:* Divide the disk into infinitesimally thin concentric rings, use the expression derived in Exercise 13.33 for the gravitational force due to each ring, and integrate to find the total force.)

Figure **P13.89**



13.90 ••• **CALC** Mass M is distributed uniformly along a line of length $2L$. A particle with mass m is at a point that is a distance a above the center of the line on its perpendicular bisector (point P in Fig. P13.90). For the gravitational force that the line exerts on the particle, calculate the components perpendicular and parallel to the line. Does your result reduce to the correct expression as a becomes very large?

Figure **P13.90**



Answers

Chapter Opening Question ?

The smaller the orbital radius r of a satellite, the faster its orbital speed v [see Eq. (13.10)]. Hence a particle near the inner edge of Saturn's rings has a faster speed than a particle near the outer edge of the rings.

Test Your Understanding Questions

13.1 Answer: (v) From Eq. (13.1), the gravitational force of the sun (mass m_1) on a planet (mass m_2) a distance r away has magnitude $F_g = Gm_1m_2/r^2$. Compared to the earth, Saturn has a value of r^2 that is $10^2 = 100$ times greater and a value of m_2 that is also 100 times greater. Hence the *force* that the sun exerts on Saturn has the same magnitude as the force that the sun exerts on earth. The *acceleration* of a planet equals the net force divided by the planet's mass: Since Saturn has 100 times more mass than the earth, its acceleration is $\frac{1}{100}$ as great as that of the earth.

13.2 Answer: (iii), (i), (ii), (iv) From Eq. (13.4), the acceleration due to gravity at the surface of a planet of mass m_P and radius R_P is $g_P = Gm_P/R_P^2$. That is, g_P is directly proportional to the planet's mass and inversely proportional to the square of its radius. It follows that compared to the value of g at the earth's surface, the value of g_P on each planet is (i) $2/2^2 = \frac{1}{2}$ as great; (ii) $4/4^2 = \frac{1}{4}$ as great; (iii) $4/2^2 = 1$ time as great—that is, the same as on earth; and (iv) $2/4^2 = \frac{1}{8}$ as great.

13.3 Answer: yes This is possible because surface gravity and escape speed depend in different ways on the planet's mass m_P and radius R_P: The value of g at the surface is Gm_P/R_P^2 while the escape speed is $\sqrt{2Gm_P/R_P}$. For the planet Saturn, for example, m_P is about 100 times the earth's mass and R_P is about 10 times the earth's radius. The value of g is different than on earth by a factor of $(100)/(10)^2 = 1$ (i.e., it is the same as on earth), while the escape speed is greater by a factor of $\sqrt{100/10} = 3.2$. It may help to remember that the surface gravity tells you about conditions right next to the planet's surface, while the escape speed (which tells you how fast you must travel to escape to infinity) depends on conditions at *all* points between the planet's surface and infinity.

13.4 Answer: (ii) Equation (13.10) shows that in a smaller-radius orbit, the spacecraft has a faster speed. The negative work done by air resistance decreases the *total* mechanical energy $E = K + U$; the kinetic energy K increases (becomes more positive), but the gravitational potential energy U decreases (becomes more negative) by a greater amount.

13.5 Answer: (iii) Equation (13.17) shows that the orbital period T is proportional to the $\frac{3}{2}$ power of the semi-major axis a. Hence the orbital period of Comet X is longer than that of Comet Y by a factor of $4^{3/2} = 8$.

13.6 Answer: no Our analysis shows that there is *zero* gravitational force inside a hollow spherical shell. Hence visitors to the interior of a hollow planet would find themselves weightless, and they could not stand or walk on the planet's inner surface.

13.7 Answer: (iv) The discussion following Eq. (13.27) shows that the difference between the acceleration due to gravity at the equator and at the poles is v^2/R_E. Since this planet has the same radius and hence the same circumference as the earth, the speed v at its equator must be 10 times the speed of the earth's equator. Hence v^2/R_E is $10^2 = 100$ times greater than for the earth, or $100(0.0339\ \text{m/s}^2) = 3.39\ \text{m/s}^2$. The acceleration due to gravity at the poles is $9.80\ \text{m/s}^2$, while at the equator it is dramatically less, $9.80\ \text{m/s}^2 - 3.39\ \text{m/s}^2 = 6.41\ \text{m/s}^2$. You can show that if this planet were to rotate 17.0 times faster than the earth, the acceleration due to gravity at the equator would be *zero* and loose objects would fly off the equator's surface!

13.8 Answer: (iii) If the sun collapsed into a black hole (which, according to our understanding of stars, it cannot do), the sun would have the same mass but a much smaller radius. Because the gravitational attraction of the sun on the earth does not depend on the sun's radius, the earth's orbit would be unaffected.

Bridging Problem

Answers: (a) Perihelion: $v_P = \sqrt{\dfrac{Gm_S}{a}\dfrac{(1+e)}{(1-e)}}$

aphelion: $v_A = \sqrt{\dfrac{Gm_S}{a}\dfrac{(1-e)}{(1+e)}}$

(b) $v_P = 54.4\ \text{km/s}$, $v_A = 0.913\ \text{km/s}$